

Non-extremal geometries and holographic phase transitions

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ABSTRACT: Using the low energy limit of type IIB superstring theory, we obtain the non-extremal limit of deformed conifold geometry which is dual to the IR limit of large N thermal QCD. At low temperatures, the extremal geometry without black hole is favored while at high temperatures, the field theory is described by non-extremal black hole geometry. We compute the ten dimensional on shell action for extremal and non-extremal geometries and demonstrate that at a critical temperature T_c there is a first order confinement to deconfinement phase transition. We compute T_c as a function of 'tHooft coupling and study the thermodynamics of the dual gauge theory by evaluating the free energy and entropy of the ten dimensional geometry. We find agreement with the conformal limit while thermodynamics of non-conformal strongly coupled gauge theories is explored using the black hole geometries in non-AdS space.

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1. Introduction

At extreme temperatures nuclear matter is best described by a weakly interacting gas of gluons and quarks due to asymptotic freedom [1, 2]. The weak coupling allows one to use perturbative techniques to study the thermodynamics of the system and at zeroth order in perturbation theory, it is best described as a gas of free particles. But the same asymptotic freedom implies that the perturbative analysis must break down with decreasing temperature. As temperature is lowered, the couplings get stronger and color degrees of freedom are confined. Thus nuclear matter undergoes a phase transition- from low temperature confined phase of color neutral constituents to high temperature deconfined phase of quarks and gluons. However to analyze the non-perturbative confinement mechanism, one has to study the theory on the lattice [3, 4, 5] or resort to effective field theories - both of which are successful along with their inherent limitations¹.

On the other hand 'tHooft showed that when number of colors N goes to infinity, only planar diagrams contribute to the amplitude [10, 11]. Thus in the large N limit, the theory drastically simplifies and one may hope that the thermodynamics could be analytically tractable. Furthermore, the structure of the planar diagrams suggest that

¹For analysis of high temperature phase and phase transitions please consult [6]-[9]. For a review of recent developments in lattice QCD, please consult [13]-[28] and [29]-[34] for the large N limit.

four dimensional gauge theory maybe interpreted as a string theory [12]. This connection becomes clear with the realization that gauge theories naturally arise from excitations of strings ending on branes [35], while on the other hand gravitons arise from low energy excitations of closed strings. By studying the interactions between open and closed strings, the correspondence between gauge theory and gravity can be realized. The best studied example is the AdS/CFT correspondence proposed by Maldacena [36] which essentially maps maximally supersymmetric $\mathcal{N} = 4$ $SU(N)$ conformal gauge theory to $AdS_5 \times S^5$ geometry. In the large N limit, the gauge theory has large 'tHooft coupling $N \rightarrow \infty, g_s \rightarrow 0, g_s N \gg 1$ and $AdS_5 \times S^5$ geometry with small curvature $\sim 1/L^2, L \equiv (g_s N)^{1/4} \sqrt{\alpha'} \gg 1$ has a classical description. Again the appearance of large N has allowed an exact analytic description of strongly coupled quantum gauge theory in terms of weakly coupled dual classical gravity.

In [37] an exact proposal was made to compute correlation functions of four dimensional $\mathcal{N} = 4$ conformal $SU(N)$ gauge theory using partition function of supergravity on AdS space. To study the thermodynamic properties of the gauge theory, one studies the thermodynamics of the dual geometry and for the case of CFTs, this amounts to the study of black holes in AdS space. The AdS/CFT correspondence has been quite successful in this regard- the scaling of thermodynamic state functions (free energy, pressure, entropy) of CFTs with respect to temperature matches with that of black holes [38, 39]. However there are no phase transitions in conformal field theories and the strong coupling regime of QCD matter is far from conformal. Thus the study of confinement to deconfinement phase transition in QCD requires careful extension of the gauge/gravity correspondence to incorporate the conformal anomalies.

In principal this can be done by placing D branes in ten dimensional geometries and then studying the warped geometry sourced by the fluxes and scalar fields arising from the branes. Since QCD is non supersymmetric and non-conformal, the first objective is to find gauge theories with RG flows arising from D brane configurations with minimal SUSY. There have been a lot of progress in that direction: In [41, 42] RG flows that connected conformal fixed points at IR and UV was incorporated, and [43] connected the UV $\mathcal{N} = 4$ conformal fixed point to a $\mathcal{N} = 1$ confining theory. But the model with QCD like logarithmic running of the coupling and minimal supersymmetry is the Klebanov-Strassler (KS) model [44] (with an extension by Ouyang [45] to incorporate fundamental matters). Although at the highest energies the gauge theory is best described in terms of bifundamental fields with effective degrees of freedom diverging, at the lowest energies the gauge theory resembles $\mathcal{N} = 1$ SUSY QCD. Again in the limit when effective $D3$ brane charge is large, the gauge theory with large 'tHooft coupling has an equivalent description in terms of warped deformed cone.

The thermodynamics of this non-conformal gauge theory is encoded in the dual geometry. A great deal of effort has been given in computing the black hole geometry dual to non-conformal thermal gauge theories. For example in [46, 47, 48, 49] the cascading picture of the original KS model was extended to incorporate black-hole without any fundamental matter, while fundamental matter was accounted for in [50]. Most of the attempts are based on obtaining an effective lower dimensional action from KK reducing ten dimen-

sional supergravity action. Dimensional reduction of a generic ten dimensional action is a formidable challenge specially when there are non-trivial fluxes and scalar fields and it is highly non-trivial to obtain a consistent truncation. Furthermore, it is not clear how RG flow of the dual gauge theory can be obtained since the fields in the effective action are not the dilaton or the flux in the original ten dimensional action.

On the other hand, partition function of the geometry and thus the thermodynamics of the dual gauge theory can be directly obtained by computing the on shell gravity action with appropriate boundary terms [51]. In a series of papers [52]-[58], we proposed the ten dimensional deformed resolved conifold black hole geometry as the dual to UV complete gauge theory which resembles large N thermal QCD. Working with the ten dimensional geometry, we avoid the difficulty of KK reduction while extracting the exact RG flow of the gauge theory. The thermal gauge theory studied in [52]-[58] has a rich phase structure and as temperature is altered, we expect phase transitions. Our goal is to study the phase transitions at strong coupling, but to do so, we must first understand how geometries can describe different phases. In this work, we make progress in that direction and obtain the most general ten dimensional geometry (with or without a black hole) that arises from low energy limit of type IIB superstring theory. Directly identifying the gauge theory partition function with that of the ten dimensional geometry, we show how phase transition is realized. For any given temperature of the dual gauge theory, there are two geometries—extremal (without black hole) and non-extremal (with black hole) but the geometry with lower on shell action is preferred. At a critical temperature T_c , both geometries are equally likely and we have a phase transition.

Our ten dimensional solution is analytic and the corrections to the metric due to the black hole can be exactly written as a Taylor series in \tilde{r}_h/r and $g_s M^2/N$ where \tilde{r}_h is the Schwarzschild horizon and M, N are number of five branes and effective number of three branes at some high energy. This series expansion allows us to write down exact expression for the on shell gravity action with or without the black hole and then obtain the critical horizon. However, in our analysis of the non-extremal geometry, we have considered constant axio-dilaton field. There are no D7 branes, no fundamental matter in the dual thermal gauge theory and no Baryochemical potential. On the other hand, it is straight forward to incorporate chemical potential in extremal ten dimensional geometry by considering gauge fluxes on holomorphically embedded D7 branes. Note that D7 brane embedding in black hole geometry is highly non-trivial in the presence of non-trivial three form fluxes². What we have been able to obtain is the black hole geometry with only non-trivial five form \tilde{F}_5 and three form G_3 but constant dilaton and already in this simplified scenario, we find a first order phase transition. Observe that the warped deformed cone of KS model is dual to $\mathcal{N} = 1$ $SU(\mathcal{M})$, $\mathcal{M} \gg 1$ pure gauge theory which has no fundamental matter but it confines, exhibits a mass gap and has gluino condensates [59]. Thus the confine-

²In a particular scaling limit, the D7 embedding in a black hole geometry with considering gauge fluxes on the D7 branes along with UV completion was considered in [57]. The form of the localized sources was proposed but the exact value of the on shell action considering all the fluxes was not determined. Hence the exact value of critical temperature was not obtained, but the scaling of the critical temperature with the number of branes should be similar to the calculation done in this paper.

ment/deconfinement phase transition we obtain should mimic the phase transition between confined glue-ball and deconfined gluons of $SU(N)$ QCD without flavor. In our upcoming work [60], we will incorporate the effect of running dilaton field and other localized sources on the non-extremal geometry which will also give the dual geometric description of UV complete thermal gauge theory.

In section 2, we briefly discuss the supergravity equations and outline the procedure of obtaining thermodynamic state functions of the gauge theory. Summarizing the extremal geometry and its dual confined gauge theory in section 2.1, we propose the non-extremal limit of ten dimensional geometry in section 2.2. Using the supergravity solutions for extremal and non-extremal geometries, in section 3, we study transitions between the geometries which essentially describes phase transitions in the dual gauge theory. When the boundary of the geometry $r = \mathcal{R}$ scales with the 'tHooft coupling $g_s N$, we find the critical temperature T_c as a function of $g_s N$. Finally in section 4, we discuss the structure of the gauge theory and propose a possible brane configuration that gives rise to the supergravity solutions. Since our ultimate goal is to learn about thermal phase transitions in nuclear matter, we describe the connections between the gauge theory and large N thermal QCD.

2. Gravity Action and Gauge Theory

We start with the type IIB supergravity action including local sources in ten dimensions [61][62]:

$$S_{\text{total}} = S_{\text{SUGRA}} + S_{\text{loc}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left(R + \frac{\partial_M \tilde{\tau} \partial^M \tilde{\tau}}{2|\text{Im}\tilde{\tau}|^2} - \frac{|\tilde{F}_5|^2}{4 \cdot 5!} - \frac{G_3 \cdot \bar{G}_3}{12\text{Im}\tilde{\tau}} \right) + \int_{\Sigma_8} C_4 \wedge R_{(2)} \wedge R_{(2)} + \frac{1}{8i\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\text{Im}\tilde{\tau}} + S_{\text{loc}} \quad (2.1)$$

in Einstein frame, where $\tilde{\tau} = C_0 + ie^{-\phi}$ is the axio-dilaton with C_0 being the axion and ϕ the dilaton field and $\tilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$ is the five-form flux sourced by the D3 or fractional D3 branes. Here $G_3 \equiv F_3 - \tilde{\tau}H_3$ with F_3 the RR three form flux and $H_3 = dB_2$ the NS-NS three form flux with B_2 being the NS-NS two form while C_2 is the RR two form. We also have C_4 the four-form potential, $G = \sqrt{-\det g_{MN}}$, $M, N = 0, \dots, 9$ with g_{MN} being the metric, $R_{(2)}$ the curvature two-form, and S_{loc} is the action for localized sources in the system (i.e. D branes).

The above action (2.1) is the most general supergravity action obtained from type IIB superstring action with fluxes and localized sources. Minimizing the action leads to the following Einstein equations

$$R_{\mu\nu} = -g_{\mu\nu} \left[\frac{G_3 \cdot \bar{G}_3}{48 \text{Im}\tilde{\tau}} + \frac{\tilde{F}_5^2}{8 \cdot 5!} \right] + \frac{\tilde{F}_{\mu abcd} \tilde{F}_\nu^{abcd}}{4 \cdot 4!} + \kappa_{10}^2 \left(T_{\mu\nu}^{\text{loc}} - \frac{1}{8} g_{\mu\nu} T^{\text{loc}} \right) \\ R_{mn} = -g_{mn} \left[\frac{G_3 \cdot \bar{G}_3}{48 \text{Im}\tilde{\tau}} + \frac{\tilde{F}_5^2}{8 \cdot 5!} \right] + \frac{\tilde{F}_{m abcd} \tilde{F}_n^{abcd}}{4 \cdot 4!} + \frac{G_m^{bc} \bar{G}_{nbc}}{4 \text{Im}\tilde{\tau}} + \frac{\partial_m \tilde{\tau} \partial_n \tilde{\tau}}{2 |\text{Im}\tilde{\tau}|^2} + \kappa_{10}^2 \left(T_{mn}^{\text{loc}} - \frac{1}{8} g_{mn} T^{\text{loc}} \right) \quad (2.2)$$

where $\mu, \nu = 0, \dots, 3$; $m, n = 4, \dots, 9$ and we have assumed that the fluxes and axio-dilaton only depends on coordinates x^m .

The equation of motion for G_3 can be expressed in terms of a seven-form $\Lambda_7 = *_{10}G_3 - iC_4 \wedge G_3$ in the following way,

$$d\Lambda_7 - \frac{i}{\text{Im}\tilde{\tau}} d\tilde{\tau} \wedge \text{Re}\Lambda_7 = 0 \quad (2.3)$$

where typically Λ_7 quantifies the deviations from the imaginary self dual (ISD) behavior. Minimizing the action (2.1) also gives the Bianchi identity for the five-form flux [62]

$$d\tilde{F}_5 = \frac{G_3 \wedge \tilde{G}_3}{2i\text{Im}\tilde{\tau}} + 2\kappa_{10}^2 T_3 \rho_3^{\text{loc}} \quad (2.4)$$

where ρ_3^{loc} is the D3 charge density from the localized sources and T_3 is the brane tension. In the presence of D3 branes, ρ_3^{loc} is a delta function peaked at the location of the branes while D7 or fractional D5 branes may also contribute to D3 charge.

To keep lorentz invariance along the space-time direction we assume the self-dual five-form has the form

$$\tilde{F}_5 = (1 + *)d\alpha \wedge dt \wedge dx \wedge dy \wedge dz. \quad (2.5)$$

where $\alpha(x^m)$ is a scalar field, function of the internal coordinates x^m . We take a general metric ansatz:

$$ds^2 = -e^{2A+2B}dt^2 + e^{2A}(dx^2 + dy^2 + dz^2) + e^{-2A-2B}\tilde{g}_{mn}dx^m dx^n$$

$$\tilde{g}_{mn}dx^m dx^n = \left(a(r)dr + k(r)ds_{\mathcal{M}_5}^2 \right) \quad (2.6)$$

where $B \neq 0$ characterizes the existence of a black hole. Using this metric ansatz we can express Λ_7 as

$$\Lambda_7 = \left[e^{4A+B} *_6 G_3 - i\alpha G_3 \right] \wedge dt \wedge dx \wedge dy \wedge dz \quad (2.7)$$

The above choice of Λ_7 leads us to three different classes of solutions from the G_3 EOM (2.3). These three classes can be tabulated in the following way:

- If $\alpha = e^{4A+B}$ in (2.7) and $\Lambda_7 = d\Lambda_7 = 0$ then G_3 must be imaginary self dual (ISD). When $B = 0$ then this is the same as GKP solution [62], and in this case $\tilde{\tau}$ is not restricted and

$$\tilde{R}_{mn} = \frac{\partial_m \tilde{\tau} \partial_n \tilde{\tau}}{2|\text{Im}\tilde{\tau}|^2} - 3\tilde{\nabla}_m \partial_n B - \frac{3}{2}\partial_m B \partial_n B \quad (2.8)$$

where \tilde{R}_{mn} is the Ricci tensor for cone metric \tilde{g}_{mn} .

- If $\alpha \neq e^{4A+B}$ then we can take $\Lambda_7 \neq 0$ but keep $d\Lambda_7 = 0$ and $d\tilde{\tau} = 0$. This means $\tilde{\tau}$ is a constant³.

³Or $\tilde{\tau} = d\lambda_{-1}$ i.e d of a (-1) -form. The functional form for the (-1) -form is non-trivial, so this option is more cumbersome to use.

- If $\alpha \neq e^{4A+B}$ then we can again take $\Lambda_7 \neq 0$ but now $d\Lambda_7 \neq 0$ and $d\tilde{\tau} \neq 0$ such that (2.3) is satisfied. This means both axion and the dilaton could run in this scenario.

The first possibility is studied in the appendix A. We are more interested in the last two options because it is well established that AdS black hole solution corresponds to conformal $SU(N)$ gauge theory, and non-AdS black hole solutions should reduce to AdS black hole solutions in certain limit. It is obvious that in the first case at $G_3 = 0$ it does not recover AdS black hole solution while the last two cases obviously do.

Suppose now we find a solution to the above set of Einstein equations (2.2) along with the flux equations (2.3), (2.4). The resulting geometry with the metric (2.6) has the topology of $Y^5 \times \mathcal{S}^5$ and in particular for $G_3 = 0$, Y^5 can be a five dimensional AdS space with or without black holes which has throat radius L while \mathcal{S}^5 can be a compact five sphere S^5 with radius L . For non-vanishing three form flux G_3 , Y^5 will be a non-AdS space and the size of \mathcal{S}^5 given by $e^{-2A-2B}k(r)$ will diverge for large r . However for fixed radial coordinate r , the geometry has topology $M_4 \times \mathcal{S}^5$ where M_4 is four dimensional Minkowski space and \mathcal{S}^5 has fixed radius. This way at fixed radial location (which we interpret as the boundary), we can obtain a four dimensional manifold M_4 by integrating over the compact space \mathcal{S}^5 . The gauge theory is defined on Minkowski space M_4 while the manifold $Y^5 \times \mathcal{S}^5$ is the dual geometry for the gauge theory.

According to AdS/CFT correspondence, the partition function of string theory on $AdS_5 \times \mathcal{S}^5$ should coincide with the partition function of $\mathcal{N} = 4$ super- Yang-Mills theory. For generalized gauge/gravity correspondence, we can identify the gauge theory partition function with that of ten dimensional dual geometry,

$$\begin{aligned} \mathcal{Z}_{\text{gauge}} &= e^{-F/T} = \mathcal{Z}_{\text{gravity}} \simeq e^{-S_{\text{gravity}}^{\text{ren}}} \\ S_{\text{gravity}}^{\text{ren}} &= S_{\text{total}} + S_{GH} + S_{\text{counter}} \end{aligned} \quad (2.9)$$

where F, T are free energy and temperature of the gauge theory, S_{total} is given by (2.1), S_{GH} is the Gibbons-Hawking boundary term [51] and S_{counter} is the counter term necessary to renormalize the action. The gravity action is evaluated on $Y^5 \times \mathcal{S}^5$ and then wick rotated $t = i\tau$ to obtain the Euclidean on shell value. The \simeq is because we have ignored all the α' corrections and loop corrections.

Using (2.9), we can in principle obtain all the thermodynamic quantities for the four dimensional gauge theory by considering the ten dimensional action $S_{\text{gravity}}^{\text{ren}}$ on shell. For instance, free energy F and internal energy E of the gauge theory are given by

$$\begin{aligned} F &= -T \log(\mathcal{Z}_{\text{gauge}}) = -T \log(\mathcal{Z}_{\text{gravity}}) = T S_{\text{gravity}}^{\text{ren}} \\ E &= T^2 \frac{\partial}{\partial T} (\log \mathcal{Z}_{\text{gauge}}) = \frac{\partial S_{\text{gravity}}^{\text{ren}}}{\partial \beta} \end{aligned} \quad (2.10)$$

where T is the temperature. Knowing the free energy one gets the pressure p and entropy s

$$\begin{aligned} p &= - \left(\frac{\partial F}{\partial V_3} \right)_T \\ s &= - \left(\frac{\partial F}{\partial T} \right)_{V_3} \end{aligned} \quad (2.11)$$

where $V_3 = \int d^3x$ is the volume of three dimensional flat space.

Our primary concern for this paper is to obtain the ten dimensional geometries that can arise from the action (2.1). However, the same action can give rise to more than one manifolds. In fact, just like the case for AdS space discussed by Hawking and Page [63] and elaborated by Witten [64], there are two manifolds $X^1 = S^1 \times M^1$ and $X^2 = S^1 \times M^2$ that minimizes the action (2.1). The manifold which has lower value for the on shell action for a given temperature of the dual gauge theory will be preferred. Since X^1 and X^2 are distinct geometries, the thermodynamics of the gauge theory will be different at different temperatures, depending on which geometry is preferred. This means X^1 and X^2 will correspond to different phases of the gauge theory and we will now analyze the manifolds in some details bellow.

2.1 Extremal geometry and confinement

The metric of the extremal geometry X^1 without any black hole, i.e. $B = 0$, is given by [44] (with Minkowski signature)

$$ds^2 = e^{2A} \left[-dt^2 + dx^2 + dy^2 + dz^2 \right] + e^{-2A} \bar{g}_{mn} dx^m dx^n \quad (2.12)$$

where $e^{-4A} = h(\rho)$ and \bar{g}_{mn} is the metric of the deformed cone

$$\begin{aligned} \bar{g}_{mn} dx^m dx^n = & \frac{1}{2} \mathcal{A}^{4/3} K(\rho) \left[\frac{1}{3K^3(\rho)} (d\rho^2 + (g^5)^2) + \cosh^2 \left(\frac{\rho}{2} \right) [(g^3)^2 + (g^4)^2] \right. \\ & \left. + \sinh^2 \left(\frac{\rho}{2} \right) [(g^1)^2 + (g^2)^2] \right] \end{aligned} \quad (2.13)$$

where \mathcal{A} is a constant, $g^i, i = 1, \dots, 5$ are one forms given by

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}, & g^2 &= \frac{e^2 - e^4}{\sqrt{2}} \\ g^3 &= \frac{e^1 + e^3}{\sqrt{2}}, & g^4 &= \frac{e^2 + e^4}{\sqrt{2}}, & g^5 &= e^5 \\ e^1 &\equiv -\sin\theta_1 d\phi_1, & e^2 &\equiv d\theta_1 \\ e^3 &\equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\ e^4 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \\ e^5 &\equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \end{aligned} \quad (2.14)$$

and

$$K(\rho) = \frac{(\sinh(2\rho) - 2\rho)^{1/3}}{2^{1/3} \sinh\rho}. \quad (2.15)$$

The three form fluxes G_3 on the deformed cone is Imaginary Self Dual (ISD) to preserve the $\mathcal{N} = 1$ supersymmetry while the five form fluxes is self dual

$$*_6 G_3 = i G_3, \quad \tilde{F}_5 = (1 + *) dh^{-1} \wedge dt \wedge dx \wedge dy \wedge dz = dC_4 + B_2 \wedge F_3. \quad (2.16)$$

Note that the metric (2.13) is Ricci flat, that is the Ricci tensor for the metric \bar{g}_{mn} denoted by $\bar{R}_{mn} = 0$ (but Ricci tensor for the metric g_{mn} denoted by $R_{mn} \neq 0$), while minimizing the action (2.1) with non-zero $\partial\tilde{\tau}$ and localized sources will give rise to an internal metric which is not Ricci flat in general. In particular, the Ricci tensor for the warped metric is

$$\begin{aligned} R_{\mu\nu} &= \eta_{\mu\nu} \frac{1}{4h(\rho)} \bar{\nabla}^2 \log h(\rho) \\ R_{mn} &= \bar{R}_{mn} - \frac{\bar{g}_{mn}}{4} \tilde{\nabla}^2 \log h(\rho) - \frac{1}{2} \partial_m \log h(\rho) \partial_n \log h(\rho) \end{aligned} \quad (2.17)$$

where

$$\bar{\nabla}^2 = \bar{g}^{mn} \partial_m \partial_n + \partial_m \bar{g}^{mn} \partial_n + \frac{1}{2} \bar{g}^{mn} \bar{g}^{pq} \partial_n \bar{g}_{pq} \partial_m \quad (2.18)$$

Now using (2.17) in (2.2) with the metric given by (2.12), one readily gets that $\bar{R}_{mn} \sim \partial_m \tilde{\tau} \partial_n \tilde{\tau} + \text{local sources}$. However, F-theory [66, 67] dictates that the running of the axio-dilation field is $\partial\tilde{\tau} \sim \mathcal{O}(g_s N_f)$ where N_f is the number of seven branes and we will only consider terms which are of $\mathcal{O}(g_s N_f)$ ignoring all higher order terms. If we also ignore the localized sources, then indeed we obtain $\bar{R}_{mn} = 0$ [57, 62].

Even though \bar{g}_{mn} remains the same the warp factor is different in different regions. In the IR region we assume the D7 branes are far away and thus the axion-dilaton is constant. The warp factor, B_2 and F_3 in this region are given in [44] and calculated to the first order. The warp factor can be written in the following form

$$h(\rho) = c_i \rho^i \quad (2.19)$$

where the coefficient c_i can be treated as constants. When doing so it is assumed that

$$*dh^{-1} \wedge dt \wedge dx^1 \wedge dx^2 \wedge dx^3 = B_2 \wedge F_3 \quad (2.20)$$

This is correct if the number of D3 branes is multiple of that of the D5 branes. In the rest of the paper we will restrict to this special case⁴.

The warped deformed conifold with $h(\rho)$ given by (2.19) for small ρ as proposed in [44] in fact removes the IR singularity of the warped conifold [65] and gives rise to linear confinement in the dual gauge theory. Unlike the regular cone, the deformed cone has a blown up S^3 at the tip of the cone and this finite size of the S^3 removes the IR divergence of the fluxes [44]. Observe that near $\rho = 0$, the metric in (2.32) reduces to that of S^3

$$\bar{g}_{mn} dx^m dx^n \sim \mathcal{A}^{4/3} (2/3)^{1/3} \left[\frac{1}{2} (g^5)^2 + (g^3)^2 + (g^4)^2 \right] \quad (2.22)$$

⁴In general

$$*dh^{-1} \wedge dt \wedge dx^1 \wedge dx^2 \wedge dx^3 = B_2 \wedge F_3 + N g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5, \quad (2.21)$$

where N is the D3 brane charge at certain energy scale r_0 where $B_2 \wedge F_3 = 0$. If N is written as $N = lM + k$ with $0 \leq k < M$, then lM can be absorbed into $B_2 \wedge F_3$ leaving only the k term. This is the cascading shown in the gravity side.

which implies that the radius of S^3 at the tip of the cone is of $\mathcal{O}(\mathcal{A}^{2/3})$. The finite size of S^3 at $\rho = 0$ gives finite value for the three form flux strength $|F_3|^2, |H_3|^2$ at the tip of the cone. This way, the IR singularity of the fluxes are removed, which consequently removes the IR singularity of the warp factor.

On the other hand, this constant \mathcal{A} modifies the embedding equation for the cone and is related to the expectation values of gauge invariant operators⁵ in the dual gauge theory [44]. Hence a non-zero \mathcal{A} results in non-zero expectation value and corresponds to spontaneous breaking of Z_{2M} group down to Z_2 where M is the number of fractional three branes placed at the tip of the cone. Note that in the absence of D7 branes, this spontaneous broken symmetry is not identical to the chiral symmetry of QCD as there are no fundamental flavor. But even in the presence of seven branes, the deformed cone metric (2.12) with warp factor (2.19) will be valid by considering the D7 branes as probes. The probe seven branes will give rise to fundamental matter and depending on their embedding, can lead to breaking of flavor chiral symmetry just like in QCD [68].

Coming back to the metric (2.13), observe that with a change of coordinates

$$r^3 = \mathcal{A}^2 e^\rho \quad (2.23)$$

for large ρ , the metric becomes

$$\bar{g}_{mn} dx^m dx^n \sim dr^2 + r^2 \left(\frac{1}{9} (g^5)^2 + \frac{1}{6} \sum_{i=1}^4 (g^i)^2 \right) \quad (2.24)$$

which is the metric of regular cone with base $T^{1,1}$. Thus only for small radial coordinate ρ , the internal metric is a deformed cone while at large ρ , we really have a regular cone with topology of $R \times T^{1,1}$.

However for large ρ , we can no longer ignore the running of the $\tilde{\tau}$ field as we will be near the seven branes. As we have a regular cone for large ρ , we can use Ouyang's holomorphic embedding of seven branes [45] to determine the running of the $\tilde{\tau}$ field along with the modified flux G_3 (which is again ISD) and \tilde{F}_5 . Then for $\rho > 0$, using change of coordinates (2.23), we obtain the warp factor

$$\begin{aligned} h(r, \theta_1, \theta_2) = & \frac{\alpha'^2}{r^4} \left(\alpha_0 + \frac{81g_s M^2}{8} \log \left(\frac{r}{r_l} \right) \left[1 + \frac{3g_s N_f}{2\pi} \left(\log \left(\frac{r}{r_{\min}} \right) + \frac{1}{2} \right) \right] \right. \\ & \left. + \frac{81g_s^2 M^2 N_f}{32\pi} \log \left(\frac{r}{r_{\min}} \right) \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right) \end{aligned} \quad (2.25)$$

where N_f is the number of D7 branes which is present in the action (2.1) and source the $\tilde{\tau}$ field while r_{\min} is the minimum radial distance reached by the seven brane. The exact brane configuration that could give rise to such a warp factor in the dual geometry will be discussed in section 4. Also α_0, r_l are constants which will be determined by matching the

⁵At the bottom of the cascade when *all* D3 branes disappear and only fractional branes remain, we have pure glue SU(M) with gluino condensates $\langle \lambda \lambda \rangle$ [59]. The dual geometry partially captures the feature of the pure glue theory and we expect non-zero \mathcal{A} corresponds to non-zero $\langle \lambda \lambda \rangle$.

above solution for large ρ with the solution (2.19) valid at small ρ . In fact α_0 is proportional to the D3 brane charge at $r = r_l$. To see this we notice that on the gravity action, we do not have any branes - only fluxes. However, due to the duality, the units of the fluxes on the gravity side should equal to the number of branes on the gauge theory side. This way the units of five form fluxes can be identified as the effective charge:

$$N_{\text{eff}} = \frac{1}{2\kappa_{10}^2 T_3} \int_{T^{1,1}} \tilde{F}_5 = -\frac{1}{216\kappa_{10}^2 T_3} \int r^5 \frac{dh}{dr} g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \quad (2.26)$$

where $T_3 = \mu_3 = (2\pi)^{-3} \alpha'^{-2}$ is the three brane tension and the Gaussian surface is warped $T^{1,1}$ with radius $\sqrt{r^2 \sqrt{h}}$ that encloses Minkowski space. Using (2.25) in (2.16), we readily get from (2.26) that N_{eff} depends on r . In particular at $r = r_l$

$$N_{\text{eff}}(r = r_l) = \frac{V_5 \alpha_0 \alpha'^2}{54\kappa_{10}^2 T_3} + \mathcal{O}(g_s M^2, g_s^2 M^2 N_f)$$

$$V_5 \equiv \int d\psi d\phi_1 d\phi_2 d\theta_1 d\theta_2 \sin\theta_1 \sin\theta_2 \quad (2.27)$$

Using the exact same argument, we get $N_{\text{eff}}(\rho = 0) = 0$ as $\partial h / \partial \rho = 0$ at $\rho = 0$ [44]. This implies that there are no effective D3 branes, they have cascaded away at the IR and we are left with only fractional D3 branes.

Note that the holomorphic D7 brane embedding of Ouyang [45] only gives $U(N_f)$ flavor symmetry group. However by considering N_f number of $D7$ and $\bar{D}7$ branes, the chiral $U_L(N_f) \times U_R(N_f)$ symmetry can be realized [68, 58] but then the warp factor (2.25) will also be modified. In addition, the warp factor (2.25) only makes sense in the large ρ region which means r_{min} is large. Thus the fundamental matter fields resulting from the holomorphic embedding have large mass and we do not expect chiral symmetry to hold for massive flavors. As supersymmetry is not broken at zero temperature for the Ouyang embedding, the warp factor (2.25) corresponds to a gauge theory with massive fundamental matter that is supersymmetric at zero temperature. Recently in [69], the dual to non-supersymmetric confining gauge theory was proposed and the metric (2.12) is consistent with this proposal with the understanding that \bar{g}_{mn} in (2.32) be replaced with the perturbed metric \bar{g}_{mn}^1 which is also Ricci flat. We will analyze possible brane configurations that may give rise to SUSY and non-SUSY gauge theories with fundamental representation matter in section 4. .

The solution (2.25) leads to divergence of N_{eff} for very large r i.e very large energies. But this is expected as the dual theory is a cascading gauge theory with effective color diverging in the UV [44, 59, 70]. However our goal is to study thermal QCD which becomes conformal and free in asymptotically large energies. Thus the dual geometry with warp factor (2.25) cannot be relevant for a QCD like theory and must be modified for $r \sim \Lambda \rightarrow \infty$. In [54, 58] a proposal was made to modify the UV dynamics of the cascading gauge theory by introducing anti five branes separated from M D5 and N D3 branes along with embedding $D7 - \bar{D}7$ branes to account for fundamental matter and chiral symmetry breaking. The additional branes modify G_3 and \tilde{F}_5 while the form of the $\tilde{\tau}$ can be obtained from F-theory [66, 67]. Note that the $D5 - \bar{D}5$, $D7 - \bar{D}7$ system behaves as dipoles and thus away

from branes, their contribution to flux and axio-dilaton is not divergent and behaves as $1/r^i, i > 0$. One can consider brane embeddings such that the fluxes and axio-dilaton fields in the dual geometry is well defined everywhere in the bulk geometry while the form of the metric (2.12) remains unchanged. However, the internal metric does not remain Ricci flat anymore and the metric (2.24) gets corrections of $\mathcal{O}(g_s^2 N_f^2)$.

The fluxes proposed in [54] essentially give $B_2 \rightarrow 0$ for $r \gg r_0$ ⁶ while the axio dilaton field behaves as $\tilde{\tau} \sim b_i/r^i, b_i \sim \mathcal{O}(g_s N_f)$ for large r . Thus for asymptotically large distances, we end up with constant axio-dilaton field and a vanishing NS-NS two form B_2 and the geometry behaves as an $AdS_5 \times T^{1,1}$. This means that addition of anti five branes have slowed down the cascade and the theory has no Landau poles and the associated UV divergences. The warp factor becomes

$$h(r, \Theta) = \frac{1}{r^4} \sum_i \left(\frac{a_i(\Theta)}{r^i} \right) \quad (2.28)$$

for large r where $\Theta \equiv (\theta_i, \phi_i, \psi)$. Using (2.26) one readily gets that $N_{\text{eff}}(r = \infty) \sim a_0$ and hence does not diverge. As N_{eff} reaches a fixed value with the geometry becoming $AdS_5 \times T^{1,1}$, the gauge theory reaches a conformal fixed point. However, for supergravity approximation to be valid, we need $N_{\text{eff}}(\infty) \sim a_0 \gg 1$ which means the 'tHooft coupling $g_s N_{\text{eff}}(\infty)$ is still very large. Although we have a conformal field theory at large scale $\Lambda = r \rightarrow \infty$, the gauge group has large number of effective colors with large 'tHooft coupling and thus the gauge theory is not asymptotically free. However, the Yang Mills coupling of our gauge theory (g_1, g_2) behaves the following way

$$\begin{aligned} \frac{1}{g_1^2} + \frac{1}{g_2^2} &= e^{-\phi} \\ \frac{1}{g_1^2} - \frac{1}{g_2^2} &= e^{-\phi} \int_{S^2} B_2 \end{aligned} \quad (2.29)$$

At asymptotically large energies, that is $r \rightarrow \infty$, $B_2 \rightarrow 0$ and we have $g_1 \simeq g_2 = g_{YM}$ with

$$\lim_{\Lambda \rightarrow \infty} \frac{2}{g_{YM}^2} = \frac{1}{g_s} \quad (2.30)$$

Now we can take the limit $g_s \rightarrow 0$ by keeping $g_s N_{\text{eff}}$ fixed, which means $g_{YM} \rightarrow 0$ at asymptotically large energies, just like in QCD. However, since $g_s N_{\text{eff}}$ is held fixed at large values, we have large 'tHooft coupling while for QCD, 'tHooft coupling goes to zero.

In summary, the extremal geometry takes the simple form given by (2.12) while the warp factor is given by (2.19, 2.25) and (2.28) for various range of the radial coordinate ρ . The fluxes for various regions of the geometry can be found in [54].

Now to obtain thermodynamic state functions such as free energy, entropy and pressure of the gauge theory that is dual to the geometry $X^1 = Y^5 \times \mathcal{S}^5$, we must obtain the on

⁶ $r_0 \sim \Lambda_0$ is related to the higgs mass in the dual gauge theory. For details of the higgsing mechanism, please consult [58]

shell supergravity action on the manifold X^1 . However note that, since there is no black hole horizon in the geometry X^1 , the Euclidean renormalized on shell action $S_{\text{gravity}}^{\text{ren}} = \beta F$ is independent of horizon with F independent of T . Then using (2.10) one obtains that the free energy is independent of temperature and (2.11) readily gives

$$s = -\frac{\partial F}{\partial T} = 0 \quad (2.31)$$

The above result for entropy is consistent with the confined phase of large N gauge theory. For confined phase, Z_N symmetry is preserved and physical quantities do not depend on the temporal radius at $\mathcal{O}(N^2)$, due to large N volume independence [71],[72]. This means free energy is independent of temperature and entropy is zero at $\mathcal{O}(N^2)$. Thus the gauge theory dual to the geometry X^1 is in the confined phase. We will later see that for X^2 , entropy is nonzero- indicating we have a deconfined phase and degrees of freedom have been released at high temperature.

2.2 Non extremal geometry

The non-extremal geometry X^2 is a regular black hole with $B \neq 0$ in (2.6). The internal manifold is typically a resolved-deformed conifold that mimics the symmetries (or breaking of symmetries) of the dual gauge theory. For simplicity, we will consider constant axio-dilaton field, that is there are no seven branes in the dual gauge theory, i.e $N_f = 0$. We make the following ansatz for the internal metric

$$\begin{aligned} \tilde{g}_{mn} dx^m dx^n &= a(r) dr^2 + k(r) \tilde{g}_{pq}^{(1)} dx^p dx^q \\ \tilde{g}_{pq}^{(1)} dx^p dx^q &\equiv \left[\frac{1}{9} (g^5)^2 + \frac{1}{6} \left(\sum_{i=1}^4 (g^i)^2 \right) \right] + \tilde{g}'_{pq} dx^p dx^q \end{aligned} \quad (2.32)$$

where g^i are defined in (2.14), $k(r) = r^2 e^{2B} b(r)$ and x^p, x^q represent angular coordinates. Note that when there are no black holes i.e. horizon $r_h = 0$ and no three form flux i.e. $M = 0$, we have $\tilde{g}'_{pq} = 0$ and $a(r) = b(r) = e^{2B} = 1$. On the other hand, as we will see explicitly, fluxes enter the Einstein equations with terms of $\mathcal{O}(g_s M^2/N)$, where

$$N \equiv \alpha_0 \simeq \frac{54 \kappa_{10}^2 T_3 N_{\text{eff}}(r = r_l)}{V_5 \alpha'^2} \quad (2.33)$$

is proportional effective number of D3 branes at certain energy scale r_l . These scalings with horizon and flux strength indicate that at the lowest order, we have

$$\begin{aligned} \tilde{g}'_{pq} &= \mathcal{O}(r_h g_s M^2/N) \\ a(r) &\equiv 1 + a_1(r) = 1 + \mathcal{O}(r_h g_s M^2/N), \quad b(r) \equiv 1 + b_1(r) = 1 + \mathcal{O}(r_h g_s M^2/N) \end{aligned} \quad (2.34)$$

Note that with $\tilde{g}'_{pq} \neq 0$ the two S^2 are squashed and topologically the base of the cone is no longer $S^2 \times S^3$.

Before solving the equations of motions, we would like to discuss some subtleties:

- We are studying the gravity dual of the gauge theory and these gravity solutions are generated by fluxes only. The gauge theory lives on N D3 branes and M D5 branes placed at the tip of a regular conifold, the back reactions of the branes will give us a warped geometry. There are fluxes in these gravity solutions whose integral $\int F$ is non-vanishing, however that does not mean there is a δ function or brane source in such solutions. Actually $\int F$ can be non-zero without any source if the volume of integrated space is non-vanishing anywhere. This is what happens in $AdS_5 \times S^5$ case, $\int_{S^5} F_5 \neq 0$ where S^5 is a 5D sphere with constant radius. Particularly in Klebanov-Tseytlin solution, the radius of the warped $T^{1,1}$ becomes zero at non-zero r and becomes ill defined at very small r , leading to singularities. This singularity was resolved by Klebanov and Strassler [44] who pointed out that at low energies, the gauge theory superpotential receives quantum corrections [73] and the corresponding dual geometry is no longer the warped regular cone but it must be warped deformed cone whose S^3 is non-vanishing at $r = 0$. Thus even though $\int_{T^{1,1}} F_5 \neq 0$ and $\int_{S^3} F_3 \neq 0$ there is no δ function in either flux equations or Einstein equations. In section (4), we will outline the brane configuration that may give rise to the dual gravity solutions that we present here.

- The Schwarzschild black hole is a vacuum solution of Einstein equations with the stress energy tensor $T_{mn} = 0$. The horizon radius is an independent parameter and not determined by the Einstein equations. By considering the weak field limit, one can obtain the geometric mass of the black hole from the horizon radius which is a free parameter. Similarly by placing D3 branes at the tip of regular cone, we can generate AdS Schwarzschild black holes and again the black hole mass is a free parameter independent of the number of D3 branes. In our solution discussed below we will find a charged black hole and we interpret the charge and mass arising from the D branes placed at the tip of the regular cone. We will see that the black hole is *not a Schwarzschild black hole* and the horizon is no longer independent of the number of D branes. However, the solution can be built from the Schwarzschild solution and thus the vacuum solution plays a crucial role in our analysis.

Now let's look at the equations of motions. We first consider the external components of the Einstein equations in (2.2). Using the form of \tilde{F}_5 (2.5), the first equation in (2.2) becomes

$$R_{\mu\nu} = -g_{\mu\nu} \left[\frac{G_3 \cdot \bar{G}_3}{48 \text{Im}\tilde{\tau}} + \frac{e^{-8A-2B} \partial_m \alpha \partial^m \alpha}{4} \right] + \kappa_{10}^2 \left(T_{\mu\nu}^{\text{loc}} - \frac{1}{8} g_{\mu\nu} T^{\text{loc}} \right) \quad (2.35)$$

On the other hand, the Ricci tensor in the Minkowski direction takes the following simple form

$$R_{\mu\nu} = -\frac{1}{2} \left[\partial_m (g^{mn} \partial_n g_{\mu\nu}) + g^{mn} \Gamma_{nM}^M \partial_m g_{\mu\nu} - g^{mn} g^{\nu'\mu'} \partial_m g_{\mu'\mu} \partial_n g_{\nu'\nu} \right] \quad (2.36)$$

where $\nu', \mu' = 0, \dots, 3$ and Γ_{nM}^M is the Christoffel symbol. Now using the ansatz (2.6) for the metric, (2.36) can be written as

$$\begin{aligned} R_{tt} &= e^{4(A+B)} \left[\tilde{\nabla}^2 (A+B) - 3\tilde{g}^{mn} \partial_n B \partial_m (A+B) \right] \\ R_{ij} &= -\eta_{ij} e^{2(2A+B)} \left[\tilde{\nabla}^2 A - 3\tilde{g}^{mn} \partial_n B \partial_m A \right] \end{aligned} \quad (2.37)$$

where the Laplacian is defined in (2.18).

The set of equations can be simplified by taking the trace of the first equation in (2.2) and using (2.37). Doing this we get

$$\begin{aligned}\tilde{\nabla}^2(4A+B) - 3\tilde{g}^{mn}\partial_n B \partial_m(4A+B) &= e^{-2A-2B} \frac{G_{mnp}\bar{G}^{mnp}}{12\text{Im}\tilde{\tau}} + e^{-10A-4B}\partial_m\alpha\partial^m\alpha \\ &+ \frac{k_{10}^2}{2}e^{-2A-2B}(T_m^m - T_\mu^\mu)^{loc}\end{aligned}\quad (2.38)$$

On the other hand using (2.35) in (2.37), one gets

$$R_t^t - R_x^x = 0 \quad (2.39)$$

which in turn would immediately imply

$$\tilde{\nabla}^2 B - 3\tilde{g}^{mn}\partial_m B \partial_n B = 0 \quad (2.40)$$

Now let's look at the Bianchi identity. Using (2.5) in (2.4) gives

$$\tilde{\nabla}^2\alpha - 3e^{-2A-2B}\partial_m B \partial^m\alpha = e^{2A-B} \frac{*_6\bar{G}_3 \cdot G_3}{12i\text{Im}\tilde{\tau}} + 2e^{-6A-3B}\partial_m e^{4A+B}\partial^m\alpha. \quad (2.41)$$

Subtracting it from (2.38) one gets the following

$$\begin{aligned}\tilde{\nabla}^2(e^{4A+B} - \alpha) &= \frac{e^{2A-B}}{24\text{Im}\tilde{\tau}} |iG_3 - *_6G_3|^2 + e^{-6A-3B} |\partial(e^{4A+B} - \alpha)|^2 \\ &+ 3e^{-2A-2B}\partial_m B \partial^m(e^{4A+B} - \alpha) + \text{local source}\end{aligned}\quad (2.42)$$

Next we consider the internal components of Einstein equations. Ricci tensor for the $x^m, m = 4, \dots, 9$ directions take the following form

$$\begin{aligned}R_{mn} &= \tilde{R}_{mn} + \tilde{g}_{mn}\tilde{\nabla}^2(A+B) - 3\tilde{g}_{mn}\tilde{g}^{\lambda k}\partial_\lambda B \partial_k(A+B) \\ &+ 3\tilde{\nabla}_m\partial_n B + \partial_m B \partial_n B - 8\partial_m A \partial_n A - 2\partial_{(m} A \partial_{n)} B\end{aligned}\quad (2.43)$$

where $\tilde{\nabla}_m$ is the covariant derivative given by

$$\tilde{\nabla}_m V_c = \partial_m V_c - \tilde{\Gamma}_{mc}^b V_b \quad (2.44)$$

for any vector V_b . Here \tilde{R}_{mn} is the Ricci tensor and $\tilde{\Gamma}_{mc}^b$ is the Christoffel symbol for the metric \tilde{g}_{mn} . Using (2.38) and (2.40) we find

$$\begin{aligned}\tilde{R}_{mn} &= -g_{mn} \frac{G_3 \cdot \bar{G}_3}{24\text{Im}\tilde{\tau}} + \frac{G_{mab} \cdot \bar{G}_n^{ab}}{4\text{Im}\tilde{\tau}} + \frac{\partial_m \tilde{\tau} \partial_n \tilde{\tau}}{2|\text{Im}\tilde{\tau}|^2} \\ &- \frac{1}{2}e^{-8A-2B}\partial_m\alpha\partial_n\alpha + 8\partial_m A \partial_n A \\ &- 3\tilde{\nabla}_m\partial_n B - \partial_m B \partial_n B + 2\partial_{(m} A \partial_{n)} B\end{aligned}\quad (2.45)$$

To solve for these equations we must first find the three-form flux. For simplicity we will ignore the running of the $\tilde{\tau}$ field, that is $d\tilde{\tau} = 0$. We further take $C_0 = 0$ and $e^\phi = g_s$.

With constant $\tilde{\tau}$ and considering fluxes on the deformed cone, $G_3 = F_3 - \tilde{\tau}H_3$ must be closed everywhere since we are ignoring all localized sources i.e. $S_{\text{loc}} = 0$ in (2.1). Then the three form flux equation becomes

$$d(e^{4A+B} *_6 G_3 - i\alpha G_3) = 0 \quad (2.46)$$

Written in terms of F_3 and H_3 , the above equation becomes

$$\begin{aligned} dC_1 &= 0, \quad C_1 \equiv e^{4A+B} *_6 F_3 - \alpha e^{-\phi} H_3 \\ dC_2 &= 0, \quad C_2 \equiv e^{4A+B} *_6 e^{-\phi} H_3 + \alpha F_3 \end{aligned} \quad (2.47)$$

where C_1 and C_2 are related by

$$*_6 C_2 = \left(\alpha - \frac{e^{8A+2B}}{\alpha} \right) *_6 F_3 + \frac{e^{4A+B}}{\alpha} C_1. \quad (2.48)$$

Our goal is to solve the system of all Einstein equation and field equations exactly. To do this we take the simplest case where we assume all the unknown functions A , B , a , b and α only depends on radial direction r . In the non-AdS extremal limit and the AdS non-extremal limit, all these functions are known, so we can use the known forms as a starting point. We use the following ansatz for the warp factors

$$\begin{aligned} e^{-4A} &\equiv h(r) = \frac{N\alpha'^2}{r^4} \left(1 + \frac{81g_s M^2}{8N} \log\left(\frac{r}{r_l}\right) + A_1(r) \right) \\ e^{2B} &\equiv g(r) = 1 - \frac{\tilde{r}_h^4}{r^4} + G(r) \end{aligned} \quad (2.49)$$

where r_l is some scale determining boundary condition of $h(r)$ and \tilde{r}_h is the Schwarzschild horizon i.e. the black hole horizon in the limit $M = 0$. We will later discuss the relation between the actual horizon r_h and \tilde{r}_h . But for now note that for F_3 to recover the Klebanov-Tseytlin solution in the limit $B = 0$, the corrections must be proportional to the black hole horizon r_h . On the other hand, these corrections must vanish as $r \mapsto \infty$. However F_3 must not depend on r since D5 brane is orthogonal to the radial direction. Thus we argue that F_3 remains the same

$$F_3 = \frac{M\alpha'}{2} g^5 \wedge \omega_2 \quad (2.50)$$

where $\omega_2 = g^1 \wedge g^2 + g^3 \wedge g^4$. On the other hand, using this form in (2.47) gives

$$\alpha H_3 = \left(\frac{3Me^\phi \alpha' e^{4A}}{2r} \sqrt{\frac{a}{b}} dr \wedge \omega_2 - e^\phi C_1 \right) \quad (2.51)$$

where we have used the ansatz (2.34),(2.49) and ignored terms of order $\mathcal{O}(r_h g_s M^3 / N^2)$, which is a good approximation in the limit $N \gg M$. If we assume $C_1 = c_1(r) dr \wedge \omega_2$, this automatically gives $dH_3 = 0$. Now from the closure of C_2 and using (2.48), we find

$$c_1(r) = \frac{3M\alpha'}{2r} \sqrt{\frac{a}{b}} \left(e^{4A} - \frac{\alpha(\alpha - \alpha(r_h))}{e^{4A+2B}} \right) \quad (2.52)$$

so we get

$$e^{-\phi} H_3 = \frac{3M\alpha' e^{-4A-2B}(\alpha - \alpha(r_h))}{2r} \sqrt{\frac{a}{b}} dr \wedge \omega_2 \quad (2.53)$$

We further compute $G_3 \cdot \bar{G}_3$

$$\begin{aligned} G_3 \cdot \bar{G}_3 dx^6 &= \frac{G_3 \wedge *_6 \bar{G}_3}{\sqrt{g_6}} \\ &= \frac{3M^2 \alpha'^2 e^{-B} (1 + \mathcal{O}(r_h g_s M^2/N))}{4r} \sqrt{\frac{a}{b}} (1 + (\alpha - \alpha(r_h))^2 e^{-8A-2B}) \sin \theta_1 \sin \theta_2 / \sqrt{g_6} dx^6 \\ \sqrt{g_6} &= \frac{e^{-6A-B} r^5 \sqrt{abb^2} (1 + \mathcal{O}(r_h g_s M^2/N)) \sin \theta_1 \sin \theta_2}{108} \end{aligned} \quad (2.54)$$

so

$$G_3 \cdot \bar{G}_3 = \frac{81M^2 \alpha'^2 e^{6A}}{r^6 b^3} (1 + (\alpha - \alpha(r_h))^2 e^{-8A-2B}) \quad (2.55)$$

where we have ignored terms of $\mathcal{O}(r_h g_s M^3/N^2)$ that arise due to the deformation of the base \tilde{g}'_{pq} .

On the other hand, $G_{mab} \bar{G}_n^{ab}$ is

$$\begin{aligned} G_{mab} \bar{G}_n^{ab} &= F_{mab} F_n^{ab} + e^{-2\phi} H_{mab} H_n^{ab} \\ &= \frac{g_{mn}}{3} (*F_3) \cdot (*F_3) - (*F)_{mab} (*F)_n^{ab} + e^{-2\phi} H_{mab} H_n^{ab} \\ &= g_{mn} \frac{27M^2 \alpha'^2 e^{6A}}{r^6 b^3 m f} - (*F)_{mab} (*F)_n^{ab} + e^{-2\phi} H_{mab} H_n^{ab} \end{aligned} \quad (2.56)$$

While $T_{mn} \equiv -(*F)_{mab} (*F)_n^{ab} + e^{-2\phi} H_{mab} H_n^{ab}$ only has following non-zero components

$$\begin{aligned} T_{rr} &= \frac{9M^2 \alpha'^2 e^{-2B}}{4r^2} \frac{a}{b} \left(e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1 \right) \frac{72}{e^{-4A} r^4 b^2} \\ &= g_{rr} \frac{162M^2 \alpha'^2 e^{6A}}{r^6 b^3} \left(e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1 \right) \\ T_{\phi_i \phi_i} &= \frac{9M^2 \alpha'^2 e^{-2B}}{4r^2} \frac{a}{b} \left(e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1 \right) \frac{6 \sin^2 \theta_i}{e^{-4A-2B} r^2 a b} \\ &= g_{\phi_i \phi_i} \frac{243M^2 \alpha'^2 e^{6A} \sin^2 \theta_i}{r^6 b^3 (2 \cos^2 \theta_i + 3 \sin^2 \theta_i)} \left(e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1 \right) \\ T_{\theta_i \theta_i} &= \frac{9M^2 \alpha'^2 e^{-2B}}{4r^2} \frac{a}{b} \left(e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1 \right) \frac{6}{e^{-4A-2B} r^2 a b} \\ &= g_{\theta_i \theta_i} \frac{81M^2 \alpha'^2 e^{6A}}{r^6 b^3} \left(e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1 \right) \end{aligned} \quad (2.57)$$

where again we have ignored terms of $\mathcal{O}(r_h g_s M^3/N^2)$ that appear due to the deformation of metric of base \tilde{g}'_{pq} .

Using the above forms of the fluxes, the only *non-trivial* internal Einstein equations⁷ (2.45) are

$$\begin{aligned}
\tilde{R}_{rr} &= g_{rr} \frac{297Q}{8} - \frac{1}{2} e^{-8A-2B} \partial_r \alpha \partial_r \alpha + 8 \partial_r A \partial_r A \\
&\quad - 3 \tilde{\nabla}_r \partial_r B - \partial_r B \partial_r B + 4 \partial_r A \partial_r B \\
\tilde{R}_{\psi j} &= -g_{\psi j} \frac{27Q}{8} + 3 \tilde{\Gamma}_{\psi j}^r \partial_r B \\
\tilde{R}_{\phi_1 \phi_2} &= -g_{\phi_1 \phi_2} \frac{27Q}{8} + 3 \tilde{\Gamma}_{\phi_1 \phi_2}^r \partial_r B \\
\tilde{R}_{\phi_i \phi_i} &= g_{\phi_i \phi_i} \frac{27(15 \sin^2 \theta_i - 2 \cos^2 \theta_i)Q}{8(2 \cos^2 \theta_i + 3 \sin^2 \theta_i)} + 3 \tilde{\Gamma}_{\phi_i \phi_i}^r \partial_r B \\
\tilde{R}_{\theta_i \theta_i} &= g_{\theta_i \theta_i} \frac{135Q}{8} + 3 \tilde{\Gamma}_{\theta_i \theta_i}^r \partial_r B
\end{aligned} \tag{2.58}$$

where $Q = M^2 \alpha'^2 e^{6A} (e^{-8A-2B} (\alpha - \alpha(r_h))^2 - 1) / (r^6 b^3 \text{Im} \tau)$ and $j = \psi, \phi_i$.

We will now solve the set of Einstein equations (2.38), (2.40) and (2.58) along with the trivial Einstein equations and the Bianchi identity (2.42). Note that with the ansatz (2.32) for the metric, the three form fluxes H_3, F_3 which satisfy the flux equation (2.46) are already determined and given by (2.50) and (2.53). Thus, once equations (2.38), (2.40), (2.42) and (2.58) are solved, we have a complete set of solution as long as the trivial Einstein equations are also solved. To proceed with the analysis, we make the following ansatz for the five form flux strength

$$\alpha = t(r) e^{4A}, \quad t(r) \equiv 1 + t_1(r) = 1 + \mathcal{O}(r_h g_s M^2 / N) \tag{2.59}$$

which is again consistent with the extremal limit (i.e. $r_h = 0$) and the AdS limit $M = 0$. The $\mathcal{O}(r_h g_s M^2 / N)$ term is exactly consistent with the ansatz for the warp factors (2.49) and the internal metric (2.32), (2.34).

Also observe that the above ansatz (2.59) along with the warp factors (2.49) reduce to the well known Klebanov-Witten black hole solutions in the limit $M = 0$. In fact, for $M = 0$, the horizon is given by \tilde{r}_h with $A_1(r) = G(r) = 0$ in (2.49) and we have a Schwarzschild black hole in $AdS_5 \times S^5$. We will refer to \tilde{r}_h as the Schwarzschild horizon while r_h denotes the black hole horizon in the presence of three form flux and other sources. With no black hole but $M \neq 0$, our ansatz reduces to the extremal limit of Klebanov-Tseytlin model with $\tilde{r}_h = r_h = B = A_1 = G = 0$. With these limiting behavior and the scaling of the warp factors, it is reasonable that the functions $A_1(r), G(r)$ appearing in (2.49) should have the following scaling

$$\begin{aligned}
A_1 &= \mathcal{O}(r_h g_s M^2 / N) \\
G &= \mathcal{O}(r_h g_s M^2 / N)
\end{aligned} \tag{2.60}$$

With the above ansatz for the warp factors and α , the system of equations can be further simplified. Using the expansions (2.49), (2.59) and (2.60), we find for $r \neq \tilde{r}_h$,

⁷By trivial Einstein equations we mean Ricci tensor, $\tilde{R}_{mn} = 0$ equations.

expanding up to linear order in $g_s M^2/N$ ⁸

$$e^{-8A-2B}(\alpha - \alpha(r_h))^2 = 1 - \frac{\tilde{r}_h^4}{r^4} + \mathcal{O}(r_h g_s M^2/N) \quad (2.61)$$

Now the Taylor series in \tilde{r}_h/r away from $r = \tilde{r}_h$ ⁹, keeping only up to linear order terms in $\mathcal{O}(g_s M^2/N)$ gives

$$\begin{aligned} g_{pq}Q &\sim \frac{\tilde{r}_h^4}{r^4} \mathcal{O}(g_s M^2/N) \\ g_{rr}Q &\sim \sum_{k=1} \frac{\tilde{r}_h^{4k}}{r^{4k+2}} \mathcal{O}(g_s M^2/N) \end{aligned} \quad (2.62)$$

But in the limit $N \gg M$, we have $\mathcal{O}(g_s M^2/N) \ll 1$ and for $\tilde{r}_h/r < 1$, we can ignore $g_{mn}Q$ all together. This drastically reduces equations in (2.58) to

$$\begin{aligned} \tilde{R}_{rr} &= -\frac{1}{2}e^{-8A-2B}\partial_r\alpha\partial_r\alpha + 8\partial_r A\partial_r A \\ &\quad - 3\tilde{\nabla}_r\partial_r B - \partial_r B\partial_r B + 4\partial_r A\partial_r B \\ \tilde{R}_{ij} &= 3\tilde{\Gamma}_{ij}^r\partial_r B \end{aligned} \quad (2.63)$$

In fact, the above equation simplifies further, with the following observation: in the limit $M = 0$, (2.63) is trivially satisfied for our ansatz of the warp factor (2.49) along with the scalings (2.60), (2.59) and (2.34). For $M \neq 0$, the right side of the equation has terms of order $\mathcal{O}(\tilde{r}_h^{4k}/r^{4k})\mathcal{O}(g_s M^2/N)$, if we ignore the deformation of the base (that is $a = b = 1, G = 0$ and $\tilde{g}'_{pq} = 0$) and consider $t_1 = A_1 = 0$. On the other hand, the left side \tilde{R}_{mn} can have terms of $\mathcal{O}(\tilde{r}_h^{4+l}/r^{4+l})\mathcal{O}(g_s M^2/N)$ if the base of the cone along with t_1, A_1, G have terms of same order. This means, to solve (2.58) everywhere from horizon to boundary along with the *trivial* internal Einstein equations, we expect the following scalings :

$$\begin{aligned} \mathcal{F}_1 &\sim \sum_{l=0}^{\infty} \left(\frac{\tilde{r}_h}{r}\right)^{4+l} \mathcal{O}\left(\frac{g_s M^2}{N}\right) \\ \tilde{g}'_{pq} &\sim \sum_{l=0}^{\infty} \left(\frac{\tilde{r}_h}{r}\right)^{4+l} \mathcal{O}\left(\frac{g_s M^2}{N}\right) \end{aligned} \quad (2.64)$$

where $\mathcal{F}_1 \equiv A_1(r), t_1(r), a_1(r), b_1(r)$ or $G(r)$. Away from the Schwarzschild horizon i.e $r > \tilde{r}_h$ such that $\mathcal{O}(g_s M^2/N) \sim \mathcal{O}(\tilde{r}_h^4/r^4) < 1$, we can ignore $\mathcal{O}(g_s M^2/N)\mathcal{O}(\tilde{r}_h^4/r^4)$ and then (2.58) is trivially solved. In fact, ignoring $A_1(r), t_1(r), a_1(r), b_1(r)$ and $G(r)$ away from

⁸The Taylor series in $g_s M^2/N$ is well defined $\forall r$ except $r = \tilde{r}_h$ and thus the expansion is valid away from but arbitrarily close to $r = \tilde{r}_h$.

⁹The Taylor series for \tilde{r}_h/r is well defined even at $r = \tilde{r}_h$, for $M \neq 0$.

the horizon gives that all the equations (2.38),(2.40),(2.58) along with the Bianchi identity (2.42) are satisfied with the following solution

$$\begin{aligned} e^{-4A} &= h_0(r) = \frac{N\alpha'^2}{r^4} \left(1 + \frac{81g_s M^2}{8N} \log \frac{r}{r_l} \right) \\ e^{2B} &\equiv g_0(r) = 1 - \frac{\tilde{r}_h^4}{r^4} \end{aligned} \quad (2.65)$$

Now what is the relation between the real horizon r_h and the Schwarzschild horizon \tilde{r}_h ? To answer this, note that since $N \gg M$, we do not expect G to be very large even at the horizon. This is also consistent with the scaling in (2.64). This means, the true horizon r_h is close to the Schwarzschild horizon \tilde{r}_h . More precisely we expect¹⁰

$$r_h = \tilde{r}_h \left(1 + \mathcal{O}(g_s M^2/N) \right) \quad (2.66)$$

Thus $r > \tilde{r}_h$ also means $r > r_h$ and hence the warp factors (2.65) with ignoring $\mathcal{F}_1, \tilde{g}'_{pq}$ works exactly away from the horizon. At the horizon, the Ricci tensor equations (2.63) are not valid and we have to incorporate the deformation of the base metric \tilde{g}'_{pq} in solving the system of equations. In fact exactly at the horizon we can see that $g_{mn}Q$ is no longer ignorable up to linear order since $\tilde{r}_h/r \sim 1$. If $\tilde{g}'_{pq} = 0$ and we consider the near horizon region, there are more than two independent equations in (2.58)- resulting in more equations than the number of independent functions i.e A_1, G, t_1, a_1, b_1 . Thus we must consider the deformations \tilde{g}'_{pq} near the horizon to solve all the Einstein equations in (2.58) and the flux equations exactly. The form of the ansatz (2.64) is precisely chosen to solve all the equations in (2.58) along with the trivial equations.

As long as we are away from the horizon, the details of the metric \tilde{g}'_{pq} and functions $A_1(r), t_1(r), a_1(r), b_1(r), G(r)$ will not effect our calculations since all these functions are second or higher order in $\gamma \sim \mathcal{O}(g_s M^2/N) \sim \mathcal{O}(r_h^4/r^4)$. Note that the flux equation (2.46) is solved exactly in the entire region $r_h \leq r \leq \infty$ up to linear order in $\mathcal{O}(r_h g_s M^2/N)$ and this equation does not involve \tilde{g}'_{pq} . On the other hand, using (2.59) in (2.51), one would obtain that H_3 will get contribution from \tilde{g}'_{pq} . But what enters into the system of equations are not just H_3 but the tensor T_{mn} and $g_{mn}G_3 \cdot \bar{G}_3$, which up to linear order and away from the horizon give rise to the simplified equation (2.63)- which does not involve \tilde{g}'_{pq} .

In summary, we have been able to obtain the exact form of flux and metric in the presence of a black hole. The three form flux is given by (2.50) and (2.53), while the scalar function that gives rise to the five form flux is given in (2.59). The modification t_1 to α from the extremal limit, the modification of the warp factor A_1, G and deviation of the internal metric is described by functions a_1, b_1 and \tilde{g}'_{pq} - all of which take the form (2.64). Thus if $r_h > \tilde{r}_h$, we have exact expressions for the metric and the fluxes in the entire region $r_h \leq r \leq \infty$, up to linear order in $\mathcal{O}(g_s M^2/N)$. Most importantly, using these solutions

¹⁰We expect $D5$ branes to increase the mass of the black hole and thus $r_h > \tilde{r}_h$. This is also consistent with our numerical analysis in [57].

and the solution for the warp factor (2.49), one can exactly compute the three form flux strength up to linear order in $\mathcal{O}(g_s M^2/N)$ - a result that allows one to compute the on shell action which will be shown in section 3.2.1.

3. Phase Transitions

Phase transitions of four dimensional $SU(N)$ gauge theory at finite temperature can be realized by spontaneous breaking of the center symmetry Z_N along the Euclidian time circle. The order parameter associated with this symmetry is the temporal Polyakov loop operator

$$W_0 = \frac{1}{N} \text{Tr } P \exp \left(i \int_0^\beta d\tau \lambda_a A_0^a \right) \quad (3.1)$$

where β is the periodicity of Euclidian time, P denotes time ordering and λ_a are the Gell-Mann matrices for $SU(N)$. In the confined phase, Z_N symmetry is preserved and $\langle W_0 \rangle$ is zero. In the deconfined phase, Z_N symmetry is spontaneously broken and $\langle W_0 \rangle \neq 0$.

In [54], we computed $\langle W_0 \rangle$ in both extremal and non-extremal geometries following the procedure of [74]-[77]. We found that for the extremal geometry, $\langle W_0 \rangle = 0$ while for the black hole geometry, $\langle W_0 \rangle \neq 0$. Although the calculation was done in UV complete geometry, it is straight forward to demonstrate that for a Klebanov-Strassler geometry with a deformed cone and constant warp factor $h(\rho) \sim \text{constant} \neq 0$ near $\rho \sim 0$, the Nambu-Goto action $S_{NG} \sim \beta F_{Q\bar{Q}} \sim \beta \mathcal{D}$, i.e. it linearly increases with inter quark separation \mathcal{D} , for large \mathcal{D} . Since $\langle W_0 W_0^\dagger \rangle \sim \exp(-\beta F_{Q\bar{Q}})$ and the free energy of a single quark is obtained in the limit $F_Q = \lim_{\mathcal{D} \rightarrow \infty} F_{Q\bar{Q}}/2$, we get $\langle W_0 \rangle \sim \exp(-\beta F_Q) = 0$. For the non-extremal geometry, even with non-UV complete scenarios, one can follow the procedure in [54] and obtain that $\langle W_0 \rangle \neq 0$.

Thus the extremal geometry X^1 corresponds to zero order parameter while black hole geometry X^2 describes non-zero order parameter. Furthermore, as already discussed in section 2.1, using (2.11) one directly obtains that X^1 corresponds to zero entropy while X^2 has black holes and non-zero entropy. Thus X^1 is the gravity dual to the confined phase of the gauge theory while X^2 describes the deconfined phase. Note that although X^1 corresponds to zero entropy, it can describe the gauge theory at non-zero temperature and we can obtain the free energy of the dual gauge theory by using (2.10) with the on shell action for X^1 . This is consistent with the understanding that confined phase can exist not only at zero temperature, but up to some non-zero critical temperature. However, at a given temperature, only one description is favored. By comparing the on shell actions for X^1 and X^2 we can obtain which geometrical description is preferred and study the Hawking-Page like phase transition between these two. But before we do that, we first study the holographic descriptions of conformal theories and demonstrate how there is no phase transitions.

3.1 Conformal field theory

For the $\mathcal{N} = 4$ Yang-Mills theory on 4D Minkowski space, there are two gravity solutions too, one is $AdS_5 \times S^5$ (X^1) and the other is black hole $AdS_5 \times S^5$ (X^2), with the Euclidean

metric in Einstein frame is

$$ds^2 = \frac{r^2}{\mathcal{L}^2} \left[\left(1 - \frac{\tilde{r}_h^4}{r^4} \right) d\tau^2 + d\vec{x}^2 \right] + \frac{\mathcal{L}^2}{r^2 \left(1 - \frac{\tilde{r}_h^4}{r^4} \right)} dr^2 + \mathcal{L}^2 d\Omega_5^2 \quad (3.2)$$

where \tilde{r}_h is the black hole horizon and $\mathcal{L}^4 = 4\pi N_c \alpha'^2$ where N_c is the number of $D3$ branes in the dual gauge theory¹¹. Note that if R, R_s are the Ricci scalar in Einstein and string frame, we have $R_s = R/\sqrt{g_s}$ with $e^\phi = g_s$ being the string coupling. Denoting string frame metric as g_{sMN} and Einstein metric as g_{MN} , we have Ricci scalar for the AdS direction as $g^{ab}R_{ab} \sim -\frac{1}{\mathcal{L}^2}$, $a, b = 0, \dots, 4$ and in the string frame $g_s^{ab}R_{sab} \sim -\frac{1}{L^2}$, $L \equiv g_s^{1/4}\mathcal{L}$. For supergravity to be a good approximation we need Ricci scalar to be small in string frame, which means we need $L^2 = \sqrt{g_s N_c} \alpha' \gg 1$. On the other hand we want to ignore string interactions, which means $g_s \rightarrow 0$. Thus the limit $\sqrt{g_s N_c} \alpha' \gg 1$ can be obtained with $N_c \rightarrow \infty$ and classical gravity with the metric (3.2) will describe the dual geometry. Observe that when $\tilde{r}_h = 0$, we recover the $AdS_5 \times S^5$ metric without any black holes.

The *bulk* on shell supergravity action for both X^1 and X^2 is the same

$$I_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left(R - \frac{|F_5|^2}{4 \cdot 5!} \right) = 0 \quad (3.3)$$

So the action difference should come from the Gibbons-Hawking (GH) term [51, 78]

$$I_{\text{GH}} = \frac{1}{\kappa_{10}^2} \int_{M_9} d^9x \sqrt{\det \tilde{g}} (D_\mu n^\mu) \quad (3.4)$$

where M_9 is the boundary at $r = \mathcal{R}$, \tilde{g} is the metric induced on M_9 , n^μ is a unit normal vector to M_9 and D_μ is the covariant derivative.

It is straight forward to compute the GH term for $AdS_5 \times S^5$ with and without black holes. We find the GH term is

$$I_{\text{GH}}^2 = \frac{\beta_2 V_3 \pi^3}{\kappa_{10}^2} \lim_{\mathcal{R} \rightarrow \infty} (4\mathcal{R}^4 - 2r_h^4) \quad (3.5)$$

for black hole $AdS_5 \times S^5$, and

$$I_{\text{GH}}^1 = \frac{4\beta_1 V_3 \pi^3}{\kappa_{10}^2} \lim_{\mathcal{R} \rightarrow \infty} \mathcal{R}^4 \quad (3.6)$$

for $AdS_5 \times S^5$ and we have used that volume of S^5 is $\int d\Omega_5 = \pi^3$.

However, the extremal geometry X^1 only describes zero temperature. The temperature described by the geometry is obtained from the periodicity of Euclidean time and for X^1 which is Poincaré AdS, this period cannot be arbitrary. This is because Poincaré AdS metric (obtained by setting $\tilde{r}_h = 0$ in (3.2)) is singular at $r = 0$. Near $r = 0$, with $V_1 = r^2/\mathcal{L}^2$ the (r, τ) part of the metric takes the following form

$$ds^2 = \frac{1}{V_1 V_1'^2} (dV_1^2 + V_1^2 V_1'^2 d\tau^2) \quad (3.7)$$

¹¹Using (2.26) with the surface integral over S^5 and $\tilde{F}_5 = (1 + *)dh^1 \wedge dt \wedge dx \wedge dy \wedge dz$, $h = \frac{\mathcal{L}^4}{r^4}$, one readily gets $N_c = \frac{\mathcal{L}^4}{4\pi\alpha'^2}$

where $V_1' = \frac{dV}{dr}$. The singularity at $V_1 = 0$ is really the singularity of the origin of polar coordinates and the metric is smooth and complete, provided $|V_1'|d\tau \equiv d\theta$, where θ is the polar angle. This readily gives that period $\beta_1 = \int d\tau = 1/|V_1'| \int d\theta \rightarrow \infty$ as $V_1 \rightarrow 0$. Thus the corresponding temperature is

$$T_1 = \beta_1^{-1} = 0 \quad (3.8)$$

On the other hand, the Euclidean black hole metric is singular at $r = \tilde{r}_h$. The singularity can be realized as that of polar coordinates with the change of variable $V_2 = \frac{r^2}{L^2} \left(1 - \frac{\tilde{r}_h^4}{r^4}\right)$ and looking at the metric near the singular point $V_2 = 0$. This way the metric is smooth and complete if and only if

$$\beta_2 = \frac{\pi \mathcal{L}^2}{\tilde{r}_h} \quad (3.9)$$

which depends on the horizon \tilde{r}_h . Also the temperature of the field theory at hypersurface $\rho = \mathcal{R}$ of geometry X^2 is now redshifted due to the presence of the black hole ¹²

$$T_2(\mathcal{R}) = \frac{\beta_2^{-1}}{\sqrt{\left(1 - \frac{\tilde{r}_h^4}{\mathcal{R}^4}\right)}} \quad (3.10)$$

In summary, the Poincaré AdS metric without a black hole only describes the vacuum phase and corresponds to zero temperature. On the other hand, for any non-zero temperature, $\mathcal{N} = 4$ SUSY gauge theory is described by the black hole geometry. Since there is no description in terms of distinct geometries for a given non-zero temperature, there is no Hawking-Page transition.

Now we would like to find the thermodynamic parameters for the $SU(N_c)$ conformal field theory. But since the black hole action I_{GH}^2 is divergent, we need to regularize it first. The divergent part $\sim \beta_2 \mathcal{R}^4$ is dependent on temperature and thus the counter term will also be dependent on it. Also observe that the regularization scheme adopted in [63, 64] identifies ΔS as the regularized gravity action. Hence regularization is not just subtraction of the infinite part $\sim \beta_2 \mathcal{R}^4$, but also a finite part arising from the vacuum action. There is one crucial difference between the case studied in [63, 64] and the Poincaré AdS geometry. Essentially, Hawking-Page considered global AdS geometry, where without a black hole, the geometry is regular and the periodicity of Euclidean time can be arbitrary. There, at a given temperature there are two descriptions, black hole and no black hole geometries and ΔS can describe the excess of energy due to presence of the black hole. For Poincaré AdS geometry, the action S^1 only describes zero temperature, so ΔS is only meaningful at zero temperature, where it is trivial. At any non-zero temperature, ΔS *does not* describe the excess energy and thus cannot be identified with the renormalized action for the black hole.

¹²The g_{tt} and g_{cc} , $c = x, y, z$ component of the Minkowski metric at hypersurface $r = \mathcal{R}$ are now distinct. The difference is dependent on \mathcal{R} and thus the local temperature at $r = \mathcal{R}$ is dependent on \mathcal{R}

However, the dual conformal thermal gauge theory has free energy that scales as $\sim T^4$ and entropy that scales as T^3 . To extract this information from our Poincaré AdS black hole geometry, special care is needed while regulatizing the action. In particular, one needs to holographically renormalize the action following the procedure of [78] and write the regularized action as a functional of the induced metric at the boundary to find the stress tensor of the dual gauge theory.

There is an alternative approach to obtain the thermodynamic state functions. Black hole geometries have entropy proportional to one quarter the surface area of the horizon [79, 80]. We can directly evaluate the black hole entropy for the ten dimensional $AdS^5 \times S^5$ black hole geometry using Wald's formula [81]-[84] where the gravity Lagrangian is $\mathcal{L}_{\text{bulk}} = \frac{1}{2\kappa_{10}^2} \left(R - \frac{|F_5|^2}{4 \cdot 5!} \right)$. The result is

$$s = \frac{2\pi^4 \tilde{r}_h^3 \mathcal{L}^2 V_3}{\kappa_{10}^2} = \frac{1}{2} N_c^2 \pi^2 V_3 T^3 \quad (3.11)$$

where $T \equiv T_2(\mathcal{R} = \infty)$ and we have used definition of $\kappa_{10}^2 = 2^6 \pi^7 \alpha'^4$. Knowing the entropy, we can of course obtain the free energy and find the appropriate counter terms that would lead to this free energy. The result is

$$\begin{aligned} F &= -\frac{1}{8} V_3 \pi^2 N_c^2 T^4 = T(I_{\text{SUGRA}}^2 + I_{\text{GH}}^2 + I_{\text{counter}}^2) \\ I_{\text{counter}}^2 &= -\frac{4\mathcal{R}^4 - 3/2\tilde{r}_h^4}{\kappa_{10}^2} \beta_2 V_3 \pi^3 \end{aligned} \quad (3.12)$$

The result for entropy and free energy is consistent with the finite temperature conformal field theory since they have the correct scaling with temperature. Also note that the entropy $s = 3/4 s_0$ where s_0 is the entropy for weakly coupled $\mathcal{N} = 4$ SYM theory. We will not discuss this discrepancy here, please refer to [38], [39] and [40].

Observe that the discussion about the phase transition about the conformal field theory is different from the case discussed in [64]. There the field theory lives on $S^1 \times S^3$, with circumference β and β' respectively. Thus the field theory is no longer conformally invariant and depends on the scale β/β' . In the large N limit there can be a phase transition as a function of β/β' .

3.2 Non-conformal field theory

Now we turn to the phase transition of non-conformal field theory. We have discussed dual gravity solutions of the non-conformal field theories in section 2. The zero-temperature solution which is discussed in section 2.1 is UV completed. The geometry is a warped deformed conifold in the IR and asymptotically AdS in the UV which ensures that the theory is holographically renormalizable. When energy flows from UV to IR as r gets smaller and smaller, the effective number of D3 brane is cascaded away, leaving only D5 branes in the IR and the theory is confined. In the previous section we discussed the finite-temperature solution but we didn't bother to UV complete it. However notice that the same AdS cap can be added to the black hole geometry with a suitable intermediate

buffering region connecting the UV and IR region. Furthermore, far away from the black hole horizon, extremal and non-extremal geometry become identical, which is evident from the form of the metric presented in the last section. At large $r \gg r_h$, $(\frac{r_h}{r})^l \ll 1$, $l > 0$ and thus the non-extremal metric for large r indeed becomes identical to the extremal limit. Hence to analyze the difference between the extremal and non-extremal geometries, we will ignore the effect of UV completions in this work and analyze how our results modify with the addition of various UV caps in our upcoming paper[60].

We will now evaluate the on-shell value of the action (2.1) for the extremal geometry X^1 without adding a UV cap and any localized sources. When the axio-dilaton field $\tilde{\tau}$ is constant, the fluxes and the metric were exactly evaluated in [44]. Thus in the absence of D7 branes such that $N_f = 0$, $\tilde{\tau} = \text{constant}$, we get for the *bulk* on shell action

$$\begin{aligned} S_{\text{total}}^1 &= \frac{1}{2\kappa_{10}^2} \left[\int d^8x \int_0^{\beta_1} d\tau \int_0^\infty d\rho \sqrt{G_1} \left(-\frac{G_3 \cdot \bar{G}_3}{24\text{Im}\tau} \right) - i \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{4\text{Im}\tau} \right] \\ &= \frac{1}{2\kappa_{10}^2} \int d^8x \int_0^{\beta_1} d\tau \int_0^\infty d\rho \sqrt{G_1} \left(\frac{5G_3^1 \cdot \bar{G}_3^1}{24\text{Im}\tilde{\tau}} \right) \\ &= \frac{1}{2\kappa_{10}^2} \int d^8x \int_0^{\beta_1} d\tau \int_0^\infty d\rho \sin\theta_1 \sin\theta_2 \left(\frac{5g_s M^2 l(\rho)}{12h_0(\rho)\text{Im}\tau} \right) \end{aligned} \quad (3.13)$$

where we have used the equations of motion for G_3 : $*_{10}G_3 = iC_4 \wedge G_3$, G_1 is the determinant of the metric (2.12) after wick rotation and $G_3^1 = F_3^1 - \frac{i}{g_s} H_3^1$, H_3^1, F_3^1 are the three form fluxes on deformed cone [44]. Also h_0 is the Klebanov-Strassler warp factor and

$$l(\rho) = \frac{\rho \coth \rho - 1}{4\sinh^2 \rho} (\sinh 2\rho - 2\rho) \quad (3.14)$$

as determined in [44]. Here $\beta_1 = T_1^{-1}$ can have any value corresponding to any temperature. This is because unlike the Poincaré AdS geometry, the deformed cone geometry (2.12) is regular and after Wick rotation, the periodicity of Euclidean time β_1 can take any value. For a chosen value of β_1 , T_1 gives the local temperature of the field theory living on the hypersurface $\rho = \rho_c$ for *any* ρ_c .¹³

Now computing the on shell *bulk* action for the non-extremal geometry X^2 gives

$$S_{\text{total}}^2 = \frac{1}{2\kappa_{10}^2} \left[\int d^8x \int_0^{\beta_2} d\tau \int_{\rho_h}^\infty d\rho \sqrt{G_2} \left(-\frac{G_3 \cdot \bar{G}_3}{24\text{Im}\tau} \right) - i \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{4\text{Im}\tau} \right] \quad (3.15)$$

where now the three from flux G_3 is distinct from the extremal values, G_2 is the determinant of the Euclidean metric and ρ_h is the horizon in the radial coordinate ρ . By switching to r coordinate using the transformation (2.23), the corresponding temperature of the field theory at hypersurface $r = \mathcal{R}$ of geometry X^2 is

$$\begin{aligned} T_2(\mathcal{R}) &= \frac{\beta_2^{-1}}{\sqrt{g(\mathcal{R})}} \\ \beta_2 &= \frac{4\pi\sqrt{h(r_h)}}{|g'(r_h)|} \end{aligned} \quad (3.16)$$

¹³The four dimensional Minkowski metric at hypersurface $\rho = \rho_c$ has identical warp factor for both the time and space directions for every ρ_c . Thus the local temperature at $\rho = \rho_c$ is independent of ρ_c .

where r_h is the horizon¹⁴ and prime denotes derivative with respect to r . The two geometries describe the same field theory at the same temperature on the hypersurface $r = \mathcal{R}$ if

$$T = T_1(\mathcal{R}) = T_2(\mathcal{R}) \Rightarrow \beta_1 = \beta_2 \sqrt{g(\mathcal{R})} \quad (3.17)$$

Also we have to take the GH term into consideration, this will be discussed in the next subsection.

Now, to see which dual geometry is favored for a gauge theory at fixed temperature T and living on hypersurface $\mathcal{R} = \infty$, we consider the action difference [63][64]

$$\begin{aligned} \Delta S &= S_{\text{total}}^2 + S_{\text{GH}}^2 - S_{\text{total}}^1 - S_{\text{GH}}^1 \\ &= \frac{\beta_2}{2\kappa_{10}^2} \int dx^8 \lim_{\mathcal{R} \rightarrow \infty} \left[\int_{\rho_h}^{\mathcal{R}} d\rho \mathcal{I}_2 - \sqrt{g(\mathcal{R})} \int_0^{\mathcal{R}} d\rho \mathcal{I}_1 \right] \\ &\quad - \frac{i}{2\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{4\text{Im}\tilde{\tau}} + S_{\text{GH}}^2 - S_{\text{GH}}^1 \\ \mathcal{I}_1 &\equiv \sin\theta_1 \sin\theta_2 \left(\frac{5g_s M^2 l(\rho)}{12h_0(\rho)\text{Im}\tilde{\tau}} \right) \\ \mathcal{I}_2 &\equiv \sqrt{G_2} \left(-\frac{G_3 \cdot \bar{G}_3}{24\text{Im}\tilde{\tau}} \right) \end{aligned} \quad (3.18)$$

where we used the relation (3.17) and $S_{\text{GH}}^1, S_{\text{GH}}^2$ are the Gibbons-Hawking surface terms for the extremal and non-extremal actions. If $\Delta S > 0$, then extremal geometry is preferred and the dual gauge theory is in the confined phase. On the other hand $\Delta S < 0$ indicates that non-extremal geometry will be favored and the gauge theory is in the deconfined phase. Note that ΔS only depends on ρ_h and thus on T . As T is altered from small to large values, it is possible for $\Delta S(T)$ to change sign. If there exists a particular $T = T_c$ such that $\Delta S(T_c) = 0$, then T_c will be the critical temperature which is realized as the confinement/deconfinement phase transition temperature of the dual gauge theory. Since we are working in the limit $N_f = 0$, the corresponding critical temperature is for pure glue large N gauge theory with no fundamental matter.

Now to evaluate (3.18) explicitly, we need to know $S_{\text{GH}}^1, S_{\text{GH}}^2, \mathcal{I}_2$ exactly, since \mathcal{I}_1 is already known. We will now compute $S_{\text{GH}}^1, S_{\text{GH}}^2$ while analyzing \mathcal{I}_2 for large and small black holes separately and estimate the critical temperature.

3.2.1 Large horizons, critical temperature and deconfined phase

When $\rho > \rho_h$ is large, the internal metric simplifies from the non-extremal limit of (2.12) and takes the form (2.32) where the radial coordinate r given by (2.23) becomes convenient. Thus for large radial distances we will use r coordinate and we really have a Klebanov-Tseytlin type geometry. Then the three form fluxes in the presence of a black hole are

¹⁴The temperature (3.16) is derived by taking the non-extremal metric to be (2.32) with $a(r)=1$. That is we are assuming the horizon is large enough such that the non-extremal limit of (2.13) takes the form (2.32).

given by (2.50),(2.53) while the flux strength is given by (2.55). Furthermore, only keeping terms up to linear order in $\mathcal{O}(g_s M^2/N)$ and using $e^\phi = g_s$, \mathcal{I}_2 can be exactly evaluated with the following result

$$\begin{aligned} \mathcal{I}_2(r) &= -\frac{r^3 g_s M^2 \left(2 - \frac{\tilde{r}_h^4}{r^4}\right) \sin\theta_1 \sin\theta_2}{32N} \\ -i \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{4\text{Im}\tilde{\tau}} &= \frac{\beta_2}{2\kappa_{10}^2} \int_{r_h}^{\infty} dr dx^8 \sin\theta_1 \sin\theta_2 \frac{3g_s M^2 r^3}{8N} \end{aligned} \quad (3.19)$$

On the other hand for large ρ , we can use the r coordinate to get

$$\mathcal{I}_1(r) = \frac{5r^3 g_s M^2 \sin\theta_1 \sin\theta_2}{16N} \quad (3.20)$$

To compute the GH term for KS geometry with black hole, we find the embedded Euclidean metric at the boundary,

$$\begin{aligned} ds_9^2 &= \frac{r^2}{\alpha' \sqrt{N \left(1 + \frac{81g_s M^2}{8N} \log \frac{r}{r_l} + A_1(r)\right)}} \left[\left(1 - \frac{\tilde{r}_h^4}{r^4} + G(r)\right) d\tau^2 + \vec{dx}^2 \right] \\ &+ \alpha' \sqrt{N \left(1 + \frac{81g_s M^2}{8N} \log \frac{r}{r_l} + A_1(r)\right)} (b(r) \tilde{g}_{pq}^{(1)} dx^p dx^q) \end{aligned} \quad (3.21)$$

where $A_1(r), G(r)$ has the form given by (2.64). The GH term is

$$S_{\text{GH}} = \frac{1}{\kappa_{10}^2} \int d^9 x \sqrt{\det \tilde{g}} g^{\mu\nu} D_\mu n_\nu = \frac{1}{\kappa_{10}^2} \int d^9 x \sqrt{\det \tilde{g}} g^{\mu\nu} (\partial_\mu n_\nu - \Gamma_{\mu\nu}^\alpha n_\alpha) \quad (3.22)$$

where \tilde{g} is the metric (3.21). Since the internal metric $\tilde{g}_{pq} = k(r) \tilde{g}_{pq}^{(1)}$ is unknown, we can not determine the GH term completely. However, from eq. (2.64) we know that the corrections to \tilde{g}_{pq} is of order $\mathcal{O}(g_s M^2/N) \frac{\tilde{r}_h^{4l}}{r^{4l}}, l > 0$, so their contributions to the GH term will be of same order. Using this scaling, we find that the GH term for KS black hole metric is

$$S_{GH}^2 = \frac{1}{108\kappa_{10}^2} \lim_{\mathcal{R} \rightarrow \infty} \left[4\mathcal{R}^4 - 2\tilde{r}_h^4 + \frac{729g_s M^2}{16N} (\mathcal{R}^4 - (1+d)\tilde{r}_h^4) \right] \beta_2 V_8 \quad (3.23)$$

where $V_8 \equiv V_3 \times V_5$. Also the term $\sim d \tilde{r}_h^4 g_s M^2/N$ comes from the $\tilde{g}'_{pq}, A_1(r)$ and $G(r)$ corrections that modifies the unit normal vector n^μ and its covariant derivative. In deriving (3.23), we only kept terms up to linear order in $\frac{g_s M^2}{N}$ which is valid when $N \gg M, N \gg \log\left(\frac{\mathcal{R}}{r_l}\right)$. When there is no black hole, i.e $r_h = \tilde{r}_h = 0$, we get the GH term for KS geometry. That is

$$S_{GH}^1 = \frac{1}{108\kappa_{10}^2} \lim_{\mathcal{R} \rightarrow \infty} \mathcal{R}^4 \left[4 + \frac{729g_s M^2}{16N} \right] \beta_1 V_8 \quad (3.24)$$

For large horizons, $\rho \geq \rho_h$ is always large and thus we can use (3.20) near the horizon to obtain the action difference

$$\Delta S = \frac{g_s M^2 \beta_2 V_8 r_h^4}{64 \kappa_{10}^2 N} \lim_{\mathcal{R} \rightarrow \infty} \left(\log \left(\frac{\mathcal{R}}{\tilde{r}_h} \right) - \frac{59}{4} - 27(d - \alpha^1) \right) \quad (3.25)$$

where α^1 arises from the following expansion

$$e^{2B} = g(r) \equiv 1 - \frac{\tilde{r}_h^4}{r^4} \left(1 + \frac{729 \alpha^1 g_s M^2}{8N} \right) + \frac{\tilde{r}_h^8}{r^8} \mathcal{O} \left(\frac{g_s M^2}{N} \right) + \dots \quad (3.26)$$

Due to the presence of D5 branes, we expect the black hole horizon to be larger than the Schwarzschild value [57], which is possible if $\alpha^1 > 0$. In the rest of the analysis, we will assume $\alpha^1 > 0$. Note that d is determined by the Einstein equations and the flux equations once boundary conditions are imposed. Since d arises from the coefficients of the term $\frac{\tilde{r}_h^4}{r^4}$ in the expansion (2.64) of $\tilde{g}'_{pq}, A_1(r), G(r)$, it is sensitive to the near horizon geometry and in particular can be obtained from the horizon values of these functions. Furthermore, d is related to $A_1(r_h)$, which in turn determines the number of effective degrees of freedom at a temperature $T \simeq r_h/L^2$. Thus from the gauge theory side, d is related to the effective colors at the thermal scale. As we are not able to exactly solve all the Einstein equations near the black hole horizon, we can speculate two scenarios:

- $d \leq -(59/108 - \alpha^1)$: In this case $\Delta S > 0$, which means extremal geometry is favored and black hole is not formed. In that case all black holes have higher free energy and for all corresponding temperatures, the gauge theory will be in the confined phase. However, this does not mean there is no phase transitions at all- rather upto linear order in $\frac{g_s M^2}{N}$, $\Delta S > 0$. In fact there are terms in ΔS which are second or higher order in $\frac{g_s M^2}{N} \log \left(\frac{\mathcal{R}}{r_l} \right)$ and for $\mathcal{R} \rightarrow \infty$, it is possible that $\frac{g_s M^2}{N} \log \left(\frac{\mathcal{R}}{r_l} \right) \gg 1$. Then the perturbative analysis breaks down and we cannot ignore higher order terms. When the higher order terms are accounted for, ΔS could indeed be zero for some r_h^c and there would be a phase transition.

- $d > -(59/108 - \alpha^1)$: In this case, using (3.25), it is possible to obtain $\Delta S = 0$ with the following value for critical horizon

$$\tilde{r}_h^c = \frac{\mathcal{R}}{\exp \left(\frac{59}{4} + 27(d - \alpha^1) \right)} \quad (3.27)$$

There is always a phase transition provided $\frac{g_s M^2}{N} \log \left(\frac{\mathcal{R}}{r_l} \right) \ll 1$. This is possible with the scaling $\mathcal{R} = N^{1/4} \sqrt{\alpha'} \rightarrow \infty$, $r_l \sim \mathcal{O}(\sqrt{\alpha'})$ and $N \gg M^2$. Then the corresponding critical temperature is

$$\begin{aligned} T_c &= \frac{1 + \mathcal{O} \left(\frac{g_s M^2}{N} \right)}{\pi \exp \left(\frac{59}{4} + 27(d - \alpha^1) \right) N^{1/4} \sqrt{\alpha'}} \\ &\sim \frac{g_s^{1/4}}{\lambda^{1/4} \sqrt{\alpha'}} \end{aligned} \quad (3.28)$$

where $\lambda \equiv g_s N$ is the 'tHooft coupling. For $T > T_c$, the black hole is formed and describes the deconfined phase which has non-zero entropy. On the other hand for $T < T_c$, we have a confined phase with zero entropy described by the extremal geometry. Thus at $T = T_c$, there is a first order phase transition in the gauge theory and we have obtained a gravitational description of it in terms of Hawking-Page phase transition between two geometries.

Now to obtain thermodynamic state functions of the gauge theory, we first need to renormalize the action. From eq. (3.19) and eq. (3.24), we see that the on shell action is divergent and we need the following counter terms:

$$S_{\text{counter}}^2 = - \lim_{\mathcal{R} \rightarrow \infty} \mathcal{R}^4 \frac{\beta_2 V_8}{\kappa_{10}^2} \left(\frac{1}{27} + \frac{59 g_s M^2}{128 N} + \frac{\kappa \tilde{r}_h^4}{54 \mathcal{R}^4} \right) \quad (3.29)$$

where κ can be determined by matching the Bekenstein-Hawking entropy of the black hole to the entropy obtained through the partition function using the relations (2.9) and (2.11). With this counter term, for $T > T_c$, we can write down the free energy and entropy for the deconfined phase of non-conformal field theory using (2.10, 2.11)

$$\begin{aligned} F &= - \frac{V_8 \tilde{r}_h^4}{2 \kappa_{10}^2} \left[\frac{1 + \kappa}{27} - \frac{g_s M^2}{32 N} \left(\log \left(\frac{\mathcal{R}}{\tilde{r}_h} \right) - \frac{32 + 27d}{2} \right) \right] \\ &\sim - N^2 T^4 \left(1 + \frac{g_s M^2 b}{N} \log(LT) \right) \\ s &\sim N^2 T^3 \left(1 + \frac{g_s M^2 b}{N} (1 + \log(LT)) \right) \end{aligned} \quad (3.30)$$

where b is some constant independent of M, N but determined by the relation between \tilde{r}_h and T . We observe that $-F/\tilde{r}_h^4 > 0$ for $N \gg M$ and increases with \tilde{r}_h . This indicates entropy increases with \tilde{r}_h . Also temperature T is given by (3.16) and since the exact metric at the horizon is not known, the exact expression for T as a function of \tilde{r}_h, M and N is also unknown. This is the reason the exact scaling of free energy with temperature is unknown and in (3.30), we approximated $T \sim \frac{\tilde{r}_h}{\pi \sqrt{N \alpha'}}$, which is a reasonable since it is the leading term and $g_s M^2/N \ll 1$. Using the same renormalization procedure for the extremal action gives $S_{\text{counter}}^1 = S_{\text{counter}}^2(\beta_1 \rightarrow \beta_2, \tilde{r}_h = 0)$ and we get $F = 0, s = 0$ for $T < T_c$ - the confined phase.

The above analysis, done purely using supergravity can be interpreted as a possible proof of deconfinement at large temperatures for a gauge theory which has a gravity dual. On the other hand, when temperature is zero we have the unique extremal geometry corresponding to the confined phase. Thus starting with a confining phase at zero temperature, our analysis shows that the gauge theory transitions to a deconfined phase provided $d > -(59/108 - \alpha^1)$ i.e the deformations of $T^{1,1}$ near the horizon behave in a particular way. Thus, we have a confinement/deconfinement phase transition and the entire analysis is done using the low energy limit of type IIB superstring action (2.1).

3.2.2 Small horizons and the confined phase

From (3.25), observe that as long as $\tilde{r}_h < \tilde{r}_h^c$, $\Delta S > 0$ and the extremal geometry without a black hole will be preferred over the black holes geometry. Thus when the horizon radius

is small compared to the critical radius \tilde{r}_h^c , the dual gauge theory will be in the confined phase.

Note that our entire analysis has been done in the large ρ region where the relevant radial coordinate is r . Going back to the ρ coordinates, we see that the critical horizon in terms of ρ coordinate becomes

$$\rho_c \sim 3 \log \left(\frac{\tilde{r}_h^c}{\mathcal{A}^2} \right) \quad (3.31)$$

The horizon $\rho_h \sim 3 \log \left(\frac{\tilde{r}_h}{\mathcal{A}^2} \right) < \rho_c$ can still be a large number referring to a large black hole. But even this ‘large’ black hole is not favored against the extremal solution since $\rho_h < \rho_c$. Thus critical horizon is the relevant scale that determines largeness and smallness.

For $\rho_h \ll \rho_c$, we do not have explicit form of the metric and the flux and thus cannot evaluate the on shell non-extremal action S^2 . Equation (3.25) does not apply in such cases and we need explicit solutions of the metric and fluxes to evaluate the action difference. Note that for these small black holes, we have to consider the non-extremal limit of (2.12), which is not known. The difficulty arises for the following reason: the proposed solution (2.32) with the scalings (2.64) is only valid for large black holes. In finding these scalings, we heavily relied on the smallness of the parameter $g_s M^2/N$. For small black holes, such scalings do not apply since at small ρ , N cascades down to smaller and smaller values. The relevant quantity now is $g_s M^2/N_{\text{eff}}(\rho)$ which is no longer small and the perturbative analysis can no longer be applied.

However, using (3.25) which is only applicable for large \tilde{r}_h i.e large ρ_h , we observe that $\Delta S = 0$ only once. On the other hand small ρ_h means small temperature and in particular at zero temperature, the $SU(M)$ pure glue dual gauge theory confines. Since there is nothing special that occurs in the gauge theory at low temperature, it is reasonable that $\Delta S \neq 0$ at low temperature. This implies ΔS becomes zero only at high temperatures and we have unique critical temperature given by the critical horizon ρ_c in (3.31). For every $\rho_h < \rho_c$, no matter how small it is, we will have confinement. Of course this is consistent with confinement at zero temperature and the scenario that deconfinement happens only at a single large temperature- indicating we have only two phases, confined and the deconfined phase. To gain a better understanding of the gauge theory and its precise degrees of freedom in the different phases, we will now sketch a possible brane configuration that gives rise to the dual geometries X^1 and X^2 .

4. Brane configurations, the gauge theory and connection to QCD

In the previous sections we studied two gravity solutions and the Hawking-Page like phase transition between them, in this section we will study the brane configurations that lead to the dual gauge theories.

Suppose we place N D3 branes at the tip of a regular cone [See Fig 1(a)]. At zero temperature, the gauge group is $SU(N) \times SU(N)$ with bi-fundamental fields $A_i, B_j, i, j = 1, 2$. This gauge theory has a conformal fixed plane and the number of D3 branes remains the same at all energy scales. Now if we put another stack of D5 branes that wraps

the vanishing two cycle at the tip of the cone [See Fig 1(b)] the gauge theory becomes $SU(N + M) \times SU(N)$ with the bi-fundamental fields, and it is no longer conformal. The $SU(M + N)$ sector has $2N$ effective flavors while the $SU(N)$ sector has $2(N + M)$ effective flavors thus it is dual to the $SU(N - M) \times SU(N)$ gauge theory under an Seiberg duality. Under a series of such dualities which is called cascading, at the far IR region the gauge theory can be described by $SU(M) \times SU(K)$ group, where $N = lM + K$, $l, 0 \leq K < M$ are positive integers. Now the number of ‘actual’ D3 branes N is no longer the relevant quantity, rather $N \pm pM$ where p is an integer describes the D3 brane charge. We take $K = 0$ in all our analysis, so at the bottom of the cascade, we are left with $\mathcal{N} = 1$ SUSY $SU(M)$ strongly coupled gauge theory which looks very much like strongly coupled SUSY QCD.

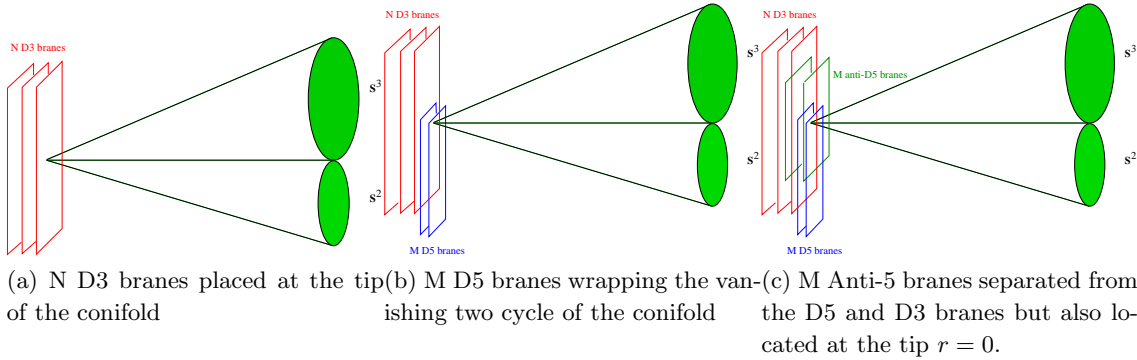


Figure 1: Brane construction of conformal field theory, non-conformal field theory and UV completed non-conformal field theory

Due to the strong coupling at the IR, the gauge theory develops non-perturbative superpotential [73] and breaks the Z_{2M} R symmetry down to Z_2 group. Since the complex fields $A_i, B_j, i, j = 1, 2$ also describe the complex coordinates of the cone, the breaking of the R symmetry modifies the geometry from a regular to a deformed cone [44]. Thus to capture the IR modification of the gauge theory, we must consider the *warped deformed cone*.

All the discussions so far have been concerned with zero temperature dynamics of the gauge theory. At finite temperature, we expect the number of D5 branes M remains constant at different temperatures, thus on the gravity side the units of F_3 fluxes remains the same too. The three form flux F_3 is given by (2.50). *This is the crucial step in our analysis since in the presence of the black hole, the base $T^{1,1}$ is modified and the spheres S^2, S^3 are now squashed and deformed. But we assume that at $r \mapsto \infty$ the base is still $T^{1,1}$ and the S^3 at the infinity still encloses the directions which the D5 branes extend along on the gauge theory side. Thus the integral of the RR flux F_3 over this S^3 should be equal to the the number of D5 branes M and F_3 is the same as in the extremal case.*

Although F_3 remains the same with or without the black hole¹⁵, H_3 given by (2.53) up to linear order in $\mathcal{O}(g_s M^2/N)$ is distinct from the extremal flux, $H_3^0 = \frac{3g_s M \alpha'}{2r} dr \wedge \omega_2$. So

¹⁵ F_3 remains the same but $\tilde{F}_3 = F_3 - C_0 H_3$ changes in the presence of the black hole, which is consistent with our earlier analysis [57].

the three form flux G_3 in the presence of the black hole is modified and it is no longer ISD. Since H_3 is modified, B_2 is also modified and this means that the change in B_2 can alter the RG flow. But as discussed in [57], this modification does not alter how the gauge theory couplings flow with energy scale, it only redefines the energy scale in terms of temperature.

Now observe that the gauge group becomes $SU(M)$ only at the far IR while at high energies, it can be described by $SU(k(\Lambda)M) \times SU((k(\Lambda) - 1)M)$ group, with $k(\Lambda)$ increasing with energy. Thus the UV of the gauge theory has divergent effective degrees of freedom and looks nothing like QCD. Although the confined phase of the gauge theory may resemble $\mathcal{N} = 1$ SUSY QCD, the deconfined phase of the gauge theory is quite different. Hence the value for critical temperature (3.28) cannot be directly related to the confinement/deconfinement temperature of QCD.

As already discussed in section 2.1, to make connections to large N QCD, we must add localized sources that can modify the UV dynamics of the gauge theory and consequently alter the large r region of the dual geometry. One possibility is to add M anti-five branes separated from each other and from the $D3 - D5$ branes at the tip of the cone. To obtain this separation, we must blow up one of the S^2 's at the tip and give it a finite size - which essentially means putting a resolution parameter. The setup is sketched in Fig. 1(c) [we did not blow up the tip, which was intentional and will be explained in what follows]. The separation gives masses Λ_0 to the $D5 - \bar{D}5$ strings and at scales less than the mass, the gauge group is $SU(N + M) \times SU(N) \times U(1)^M$ where the additional $U(1)$ groups arise due to the massless strings ending on the same $\bar{D}5$ brane. At scales much larger than Λ_0 , $D5 - \bar{D}5$ strings are excited and we have $SU(N + M) \times SU(N + M)$ gauge theory. For $\Lambda < \Lambda_0$, i.e. at low energy, gauge theory is best understood as arising from the set up of Fig. 1(b) (since the modes from $\bar{D}5$ branes are not excited). At high energy $\Lambda \gg \Lambda_0$, the gauge theory is best described as arising from Fig. 1(c). Essentially at high energies, the resolution $r_0 \sim \Lambda_0$ is not significant and we have effectively $N + M$ D3 charge at the tip of the regular cone, which arises by placing M number of $D5 - \bar{D}5$ pairs at the tip.

Of course there will be tachyonic modes arising from $D5 - \bar{D}5$ strings and one needs to consider electric and magnetic fluxes on the respective branes to stabilize the system [85]-[91]. Furthermore, to incorporate fundamental matter and chiral symmetry breaking, we need to add $D7 - \bar{D}7$ branes to the brane configuration of Fig. 1 and the details were discussed in [58]. The RG flow arising from a UV complete model with $D3, D5$ and anti $D5$ branes is sketched in the Fig.2.

On the supergravity side, the anti five branes will source RR and NS-NS three form flux such that the total flux $F_3, H_3 \rightarrow 0$ as $r \rightarrow \infty$. As already discussed in section 2.1, vanishing of the fluxes indicate that the gauge theory reaches conformal fixed point where the two gauge couplings g_1, g_2 becomes identical. At the far UV, since both the groups have same rank, we effectively have a single Yang Mills coupling $g_{YM} \rightarrow 0$ with 'tHooft coupling $g_{YM}N_{\text{eff}}$ held fixed at large value. Thus this UV completion arising from anti brane sources in principle gives rise to a QCD like theory which confines in the IR and becomes conformal in the UV. In fact the Yang Mills coupling of the gauge theory becomes free in the far UV- just like QCD, indicating that the gauge theory is indeed very similar to large N QCD.

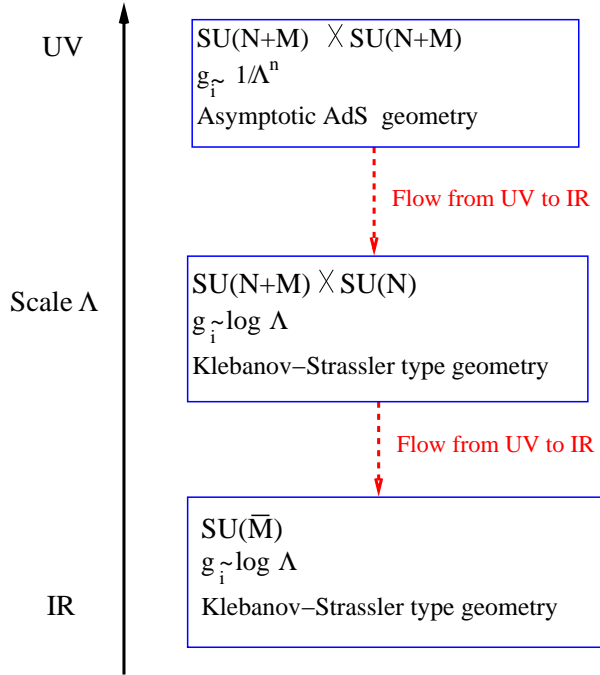


Figure 2: Gauge theory arising from the brane configuration in Fig. 1(c)

5. Conclusions

In this paper we studied the non-extremal geometries on the dual gravity side of the non-conformal finite temperature field theory. To find the exact non-extremal geometry analytically is extremely difficult because the solution is no longer supersymmetric and we have no control over the internal manifold. Even numerically it will be a formidable task to solve all the Einstein equations and field equations together. However up to linear order in $O(g_s M^2/N)$, the supergravity equations drastically simplify and the deformations of the internal metric along with the corrections to black hole factor and the warp factor can be evaluated as an infinite Taylor series in $\frac{r_h}{r}$. Using this expansion, it is straight forward to evaluate the on shell ten dimensional gravity action and we find a phase transition between the extremal and non-extremal geometry. Since the geometry is dual to a non-conformal gauge theory, this transition is interpreted as the confinement/deconfinement phase transition.

The non-extremal gravity solution we discussed is not UV completed. However, under the scaling $\mathcal{R} = N^{1/4}\sqrt{\alpha'} \rightarrow \infty$ and $N \gg M$, we get $\frac{g_s M^2}{N} \log \frac{\mathcal{R}}{r_l}$ to be small and there is no logarithmic divergence. In fact there are no $\log(\mathcal{R})$ divergence in the Gibbons-Hawking boundary term. The only term that has logarithmic term arises from the bulk action- but since $\tilde{r}_h > \tilde{r}_h^c$ scales with \mathcal{R} , the term $\log \frac{\mathcal{R}}{\tilde{r}_h}$ is in fact convergent. The critical temperature of the confinement/deconfinement we find in section 3 is obtained using this particular scaling and thus our analysis is somewhat restrictive. It will be interesting to UV complete the geometry and then there will be no need for this particular scaling since there will no

logarithmic divergences to begin with. Using the UV complete geometry to study the phase transition and other thermal properties of the non-conformal field theory is the content of our upcoming work [60].

We also did not include effect of running dilaton in the black hole geometry and did not consider D7 branes and other localized sources in non-extremal dual geometry. This means the thermodynamics we obtain is strictly restrictive to gauge theory with no flavor, no Baryochemical potential and it is not surprising that we obtain a first order phase transition. However the perturbative procedure outlined here to solve the Einstein equations along with the flux equations can easily be generalized in the presence of a running dilaton field and other localized sources. Similarly the on shell action in the presence of running axio-dilaton fields and localized sources can also be evaluated up to linear order in our perturbative parameter. Note that for a UV complete scenario, we must replace $M \rightarrow M(r)$, $\lim_{r \rightarrow \infty} M(r) \rightarrow 0$ and thus the on shell value of the action will be significantly different. In fact, the presence of localized sources will dictate how fast $M(r)$ vanishes- which also indicates how rapidly the theory becomes conformal. This means the width of the conformal anomaly will be highly sensitive to the details of the localized sources. In [58] a detailed gauge theory analysis was done and the non-extremal dual geometry will be presented in [60]

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A. Appendix

In this appendix we study $\alpha = e^{4A+B}$ case in section 2, assuming the internal 5D manifold \mathcal{M} is $T^{1,1}$ and $G_3 = 0$.

By simplifying the Einstein equations and flux equations we finally get the following four equations,

$$\begin{aligned}
k^5 e^{-4A+B} (4A' + B') &= 1 \\
2B'' + 4B'^2 + 5\frac{k''}{k} - \frac{a'}{a}B' - \frac{5}{2}\frac{a'}{a}\frac{k'}{k} + 10\frac{k'}{k}B' &= 0 \\
2e^{2B}akk'' + 8e^{2B}ak'^2 + 4e^{2B}akk'B' + 4e^{2B}ak^2B'^2 \\
-e^{2B}a'k^2B' - e^{2B}a'kk' - 8a^2 + 2e^{2B}ak^2B'' &= 0 \\
\frac{a'}{a} - 10\frac{k'}{k} - 4B' - 2\frac{B''}{B'} &= 0
\end{aligned} \tag{A.1}$$

After further simplification we find,

$$\begin{aligned}
k &= \frac{1}{(1 - e^{2B})^{1/4}}, \quad a = e^{2 \int \frac{dk'}{k}}, \\
B' + 2 - 2e^{-2B} &= 0, \quad \text{or} \quad B'^2(1 + 4e^{-2B}) = 2B''(1 - e^{-2B})
\end{aligned} \tag{A.2}$$

The first differential equation of B gives $e^{2B} = 1 - 1/r^4$ which is the usual AdS black hole solution. The second differential equation of B can be simplified as

$$B' = e^B(-1 + e^{-2B})^5/2 \quad (\text{A.3})$$

It does not have an analytic expression, but we can see that $e^{2B} \leq 1$. When $B \ll 0$ eq. (A.3) can be approximately solved $e^{2B} \simeq 4r$. Once B is known we can easily get A , a and k .

Now if G_3 is non-vanishing it should be ISD as in the GKP paper, and thus it does not effect the internal manifold at all, the only thing that changes is the Bianchi identity of the five form flux, and thus the warp factor A . So we have another solution with the same a , k and B but different A .

We are not very sure about the use of this kind of solutions or whether they are gravity duals to some field theory. It might be interesting to get numerical solutions and explore the properties of them such as stability, KK reductions, etc.

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